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Cheeger Constants of Two Related Hyperbolic Riemann Surfaces

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Abstract

This thesis concerns the study of the Cheeger constant of two related hyperbolic Riemann surfaces. The first surface R is formed by taking the quotient $\mathbb{U}^2/\Gamma(4)$, where $\mathbb{U}^2$ is the upper half-plane model of the hyperbolic plane and $\Gamma(4)$ is a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, an isometry group of $\mathbb{U}^2$. This quotient is shown to form a Riemann surface which is constructed by gluing sides of a fundamental domain for $\Gamma(4)$ together according to certain specified side pairings. To form the related Riemann surface R', we follow a similar procedure, this time taking the quotient $\mathbb{U}^2/G$, where G is an index 2 subgroup of $\Gamma(4)$. For both R and R', we provide an estimate of the Cheeger constant using a procedure given in [2]. The Cheeger constant is believed to be the same for both surfaces.

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1 Background

In this section we introduce the necessary background material needed in order to state and answer the main question of this thesis. This includes an introduction to hyperbolic geometry, a discussion of pertinent topological and algebraic notions, and a listing and verification of a handful of facts about a particular isometry group $\Gamma(4)$.

1.1 Hyperbolic Space

Here we introduce the space in which we will be working, the hyperbolic plane. We also introduce the metric, geodesics, and isometries of the space as they will play an important role throughout. We conclude with a discussion of isometric circles of certain isometries and introduce a metric which may be placed on the particular isometry group we will be studying.

The hyperbolic plane $\mathbb{H}^2$ can be introduced in relation to the usual Euclidean plane, as the points in it obey many of the same axioms as that of the Euclidean plane. The main difference between the two comes from the axiom of parallelism. This states that, in a given plane, given a line and a point not on that line, there is one and exactly one line passing through the given point that does not intersect the given line. It is the distinction of "one and only one" that is equivalent to Euclid's fifth postulate and determines that we are working in Euclidean geometry. If we change this phrase to read either "none" or "infinitely many", then we have entered into a geometry that is distinctly non-Euclidean. It is the later of these two, called "hyperbolic" geometry, in which we work. There are various models in the Euclidean plane used to represent the hyperbolic plane, including the Klein model, the Poincaré disc model, and the Poincaré upper half-plane model. The last, which we call the upper half-plane for short, is what we use throughout. To be explicit, this is the set $\mathbb{U}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\} \subset \mathbb{C}$. In this model, the boundary $\partial\mathbb{U}^2$ is the real line together with the point at infinity formed by extending the complex plane. That is, $\partial\mathbb{U}^2 = \{z = x + iy \in \mathbb{C} \mid y = 0\} \cup \{\infty\}$.

There are many equivalent metrics that we may adopt for different models of $\mathbb{H}^2$ (cf. [1], p. 130). Of these, we will use the metric $d(z_1, z_2) = \cosh^{-1}\left(1 + \frac{(x_2-x_1)^2+(y_2-y_1)^2}{2y_1y_2}\right)$, where $z_1, z_2 \in \mathbb{U}^2$. As a result, the *geodesics*, or shortest paths, of this space are the Euclidean vertical lines and semicircles whose centers lie on the real axis. As we refer to these geodesics often by their endpoints on the real axis, let us establish some notation for them. Let $[a, b]$ denote the geodesic with endpoints $(a, 0)$ and $(b, 0)$. Additionally, if the geodesic is a Euclidean vertical line given by $x = c$, we will denote it $[c, \infty]$.

The set of *isometries*, or distance preserving mappings, of the space are a set of Möbius transformations of $\hat{\mathbb{C}}$. That is, the mappings of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. By the identification

$$\frac{az + b}{cz + d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we see that this set of Möbius transformations is in one-to-one correspondence with the set of 2x2 matrices $\mathrm{PGL}_2(\mathbb{R})^+$. That is, the general linear group of 2x2 matrices with real entries and positive determinant, where two matrices are identified together if one is a scalar multiple of the other. The determinant of each matrix is taken to be positive so that $\mathbb{U}^2$ is preserved This is the group of isometries we begin to consider.

For the purposes of this research, we consider a very specific subgroup of $\mathrm{PGL}_2(\mathbb{R})^+$. To introduce this subgroup, we first consider the set of matrices with determinant 1 having integer entries. That is,the projective special linear group $\mathrm{PSL}_2(\mathbb{Z})$. There is a homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ induced by reducing entries modulo n, for some $n \in \mathbb{N}$. The *principal congruence subgroup of level* n, $\Gamma(n)$, is defined to be the kernel of this homomorphism. Explicitly, $\Gamma(n) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{n}, b \equiv c \equiv 0 \pmod{n}\}$. For this research, we will be considering the principal congruence subgroup of level 4, $\Gamma(4)$.

It is possible to describe the isometries of $\Gamma(4)$, and, more generally, any group of Möbius transformations, by the points the isometries fix in $\mathbb{U}^2$ (or on the boundary $\partial\mathbb{U}^2$). We make the following three classifications: The *elliptic* elements of $\Gamma(4)$ are those elements which fix exactly one point inside $\mathbb{U}^2$. The *parabolic* elements of $\Gamma(4)$ are those elements which fix exactly one point on the boundary of $\mathbb{U}^2$. Lastly, the *hyperbolic* elements of $\Gamma(4)$ are those which fix exactly two points on the boundary of $\mathbb{U}^2$. We shall see in a future section that we may identify an isometry as elliptic, parabolic, or hyperbolic by the trace of the corresponding matrix. Finally, we mention that each hyperbolic element, in fixing two points on the boundary of $\mathbb{U}^2$, also fixes the geodesic beginning and terminating at these fixed points. This geodesic is called the *axis* of the hyperbolic element.

Let us determine how these fixed points behave under conjugation by various elements of the isometry group. Let G be an isometry group of $\mathbb{U}^2$ and let $g \in G$ be an elliptic element fixing the point $w \in \mathbb{U}^2$. If we conjugate g by some other element $h \in G$, then the fixed point of hgh^{-1} is $h(w)$, since $(hgh^{-1})(h(w)) = h(g(w)) = h(w)$. That is, if g fixes w, then g conjugated by h fixes $h(w)$. A similar argument holds if g is parabolic or hyperbolic. So, conjugation by a group element g has the effect of moving the fixed points of an isometry. Also, in the event that the isometry is hyperbolic, conjugation by a group element moves the axis of the isometry as well. This is because if g fixes z and z', and hence the geodesic between them, then hgh^{-1} fixes $h(z)$ and $h(z')$, and hence also the geodesic between them. Finally, we mention that, as it is the kernel of the homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \mapsto \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$, the isometry group $\Gamma(4)$ is normal in $\mathrm{PSL}_2(\mathbb{Z})$. Thus, we may conjugate elements of $\Gamma(4)$ by elements of $\mathrm{PSL}_2(\mathbb{Z})$ to obtain group elements that again belong to $\Gamma(4)$.

Next, we come to the notion of an isometric circle of a Möbius transformation. For this, let g be the Möbius transformation associated with the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $c \neq 0$. Geometrically, the *isometric circle* of g is the set of points of $\hat{\mathbb{C}}$ for which g acts as a Euclidean isometry in addition to acting as a hyperbolic isometry. We shall soon see why the name "circle" is a proper one. As a set, the isometric circle of g is the set $Q_g = \{z \in \mathbb{C} : |cz + d| = |ad - bc|^{\frac{1}{2}}\}$ (cf. [1], p. 57). To verify that g acts as a Euclidean isometry on these points, let $z, w \in Q_g$. Then we have $|g(z) - g(w)| = \left|\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}\right| = \left|\frac{(ad-bc)(z-w)}{(cz+d)(cw+d)}\right| = \left|\frac{(ad-bc)(z-w)}{(ad-bc)^{\frac{1}{2}}(ad-bc)^{\frac{1}{2}}}\right| = |z-w|$. If we suppose further that $g \in \Gamma(4)$, then $ad - bc = 1$ so that $Q_g = \{z \in \mathbb{C} : |cz + d| = 1\}$. We may rewrite $|cz + d| = 1$ as $|z + \frac{d}{c}| = \frac{1}{|c|}$, so that we may readily see that Q_g is a circle in $\hat{\mathbb{C}}$ centered at $(-\frac{d}{c}, 0)$ with radius $\frac{1}{|c|}$. This is why we assumed $c \neq 0$, so that this circle has finite radius. The case in which $c = 0$ is addressed in the next paragraph. If we restrict $\hat{\mathbb{C}}$ to $\mathbb{U}^2$, then these isometric circles become semicircles, still centered at $(-\frac{d}{c}, 0)$ and having radius $\frac{1}{|c|}$. Moreover, as $g \in \Gamma(4)$, $c = 4n$ and $d = 4m + 1$ for some $m, n \in \mathbb{Z}$, where $n \neq 0$. So, $-\frac{d}{c} = \frac{4m+1}{4n}$ and $\frac{1}{|c|} = \frac{1}{4|n|}$. The "largest" isometric circles occur when $|n| = 1$, in which case they are semicircles centered at such values as $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, etc. with radii all equal to $\frac{1}{4}$. Thus, these are the geodesics of $\mathbb{U}^2$ whose endpoints lie on the real axis and have consecutive half-integer values. If $|n| > 1$, then the isometric circles have centers on the real axis and radii less than $\frac{1}{4}$ so that every other isometric circle of this form is contained in the previously mentioned "largest" isometric circles.

To develop the notion of isometric circles for those matrices in which $c = 0$, we turn to §7.36 in [1]. Here, we have the definition that, for a non-trivial isometry g of the hyperbolic plane, the isometric circle I_g is the set $I_g = \{z : \rho(z, 0) = \rho(z, g^{-1}(0)\}$, where ρ is the hyperbolic metric and 0 denotes the origin. From this definition, we see that I_g is the *perpendicular bisector* of 0 and $g^{-1}(0)$, which is the set of points equidistant from 0 and $g^{-1}(0)$. Let us see how this formulation is helpful in determining the isometric circles of elements of $\Gamma(4)$ for which $c = 0$. Let g be the isometry corresponding to the matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma(4)$. As $\Gamma(4) \in \mathrm{PSL}_2(\mathbb{Z})$, $ad - (0)c = ad = 1$ and $a, d \in \mathbb{Z}$. Hence, $a = d = \pm 1$. Moreover, $b = 4n$ for some $n \in \mathbb{Z}$. so, g corresponds to the matrix $\left(\begin{smallmatrix} 1 & 4n \\ 0 & 1 \end{smallmatrix}\right)$ and is thus translation by $4n$. Then, 0 is sent to $(-4n, 0)$ by g^{-1}, and so the isometric circle of g is given by the bisector of 0 and $(-4n, 0)$. The set of points equidistant to these points is the geodesic given by $x = \frac{-4n}{2} = -2n$, which is a Euclidean vertical line. So, the isometric circles of g and g^{-1} consist of a pair of parallel (in both the hyperbolic and Euclidean sense) lines whose Euclidean distance is $4n$. If $|n| = 1$, then we find that the "largest" isometric circles of this form are the closest Euclidean vertical lines distance 4 apart.

Briefly, we also mention the geometric notion of a horocycle. While a more general definition exists, we shall only concern ourselves with the notion of a horocycle in $\mathbb{U}^2$. A *horocycle based at* $w \in \partial\mathbb{U}^2$ is the circle tangent to $\partial\mathbb{U}^2$ at the point w that is orthogonal to each of the geodesics emanating from w. When w is a point on the real axis, these horoballs look like Euclidean circles lying in $\mathbb{U}^2$ and tangent to the real axis at w. Lastly, a *horoball* is the interior of a horocycle.

Finally, as it will be useful later, we close this section with a discussion of a metric that can be placed on $\Gamma(4)$, and even more generally, on $\mathrm{PGL}_2(\mathbb{R})^+$. To begin, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{R})^+$, and define the norm of A as $\|A\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$. It is verified (cf. [1], p. 12) that this is indeed a norm on $\mathrm{PGL}_2(\mathbb{R})^+$ and moreover that $\|A - B\|$, $A, B \in \mathrm{PGL}_2(\mathbb{R})^+$, is an induced metric for $\mathrm{PGL}_2(\mathbb{R})^+$.

1.2 Topological and Algebraic Notions

The group $\Gamma(4)$ has many interesting properties, which we note in a future section. To understand many of these properties, we need some additional topological and algebraic notions, which we introduce here. We begin with a handful of definitions, all from [1].

A *topological group* G is both a group and a topological space, the two structures being related by the requirement that the maps $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ are continuous. If the space G is endowed with the discrete topology, then G is a *discrete topological group*. In [1], Beardon gives a useful condition for a topological group to be discrete, which is presented as Corollary 1.5.2. We reproduce it here.

Corollary 1.2.1. *Let G be a topological group such that for some $g \in G$, the set $\{g\}$ is open. Then each set $\{y\}$ $(y \in G)$ is open and G is discrete.*

Thus, if we can prove that some singleton set in our topological group is open, then we have that the group is discrete.

Next, we move to the notion of a discontinuous group action. For this, let X be a topological space and G a group of homeomorphisms of X onto itself. Then, G *acts discontinuously on* X if and only if for every compact subset K of X, $g(K) \cap K = \emptyset$, except for a finite number of $g \in G$. Essentially, for most g in G, the subset K will be moved entirely off of itself by the action of g. We also want to introduce the notion of invariance under a group action. For this, let us relax our above conditions so that X is some set and G is a group that acts on it. Let $Y \subset X$. We say that Y is *invariant under the action of* G if, for all $g \in G$, $g(Y) \subset Y$. That is, Y is mapped onto itself under the action of g, though g may not fix every element of Y.

The preceding definitions have been leading to an important theorem regarding Riemann surfaces. Briefly, a *Riemann surface* is a topological space that, locally, has the same structure as the complex plane. With this, we can present the following theorem (Theorem 6.2.1 in [1]):

Theorem 1.2.2. *Let D be a subdomain of $\hat{\mathbb{C}}$ and let G be a group of Möbius transformations which leaves D invariant and which acts discontinuously in D. Then D/G is a Riemann surface.*

Note that this theorem alone does not indicate any additional structure of the Riemann surface formed (cusps, punctures, genus, etc.). One way to determine such additional structure is to determine more information about the structure of the group G. Thus, we introduce some additional definitions below regarding additional group structures. In particular, we introduce the notion of a Fuchsian group and some important properties about Fuchsian groups.

A *Fuchsian group* G is a discrete subgroup of the group of Möbius transformations with an invariant disc D. As Beardon explains, in general we may take the hyperbolic plane to be G-invariant so that we may consider G to be a discrete group of isometries of the hyperbolic plane. Each Fuchsian group G admits a fundamental set. A *fundamental set* for G is a subset F of $\mathbb{H}^2$ which contains exactly one point from each orbit in $\mathbb{H}^2$. Next, we have the definition of a fundamental domain (Definition 9.1.1 in [1]).

Definition 1.2.3. A subset D of the hyperbolic plane is a *fundamental domain* for a Fuchsian group G if and only if

(1) D is a domain;

(2) there is some fundamental set F with $D \subset F \subset \bar{D}$;

(3) h-area$(\partial D) = 0$.

To better understand the structure of the fundamental domain for G, we introduce the notion of a Dirichlet polygon (cf. [1], p.226). Given a Fuchsian group G acting on $\mathbb{U}^2$ and a point $w \in \mathbb{U}^2$ which is not fixed by any elliptic element of G, consider the following sets: $L_g(w) = \{z \in \mathbb{U}^2 : \rho(z, w) = \rho(z, gw)\}$ and $H_g(w) = \{z \in \mathbb{U}^2 : \rho(z, w) < \rho(z, gw)\}$. We see that $L_g(w)$ is the perpendicular bisector of w and gw and is thus a geodesic not containing w, and that $H_g(w)$ is the half-plane determined by $L_g(w)$ which contains w. We then have the following definition (Definition 9.4.1 in [1]):

Definition 1.2.4. The *Dirichlet polygon* $D(w)$ for G with *center* w is defined by

$$D(w) = \bigcap_{g \in G, g \neq I} H_g(w).$$

This definition is followed by a brief discussion and a theorem that states that $D(w)$ serves as a fundamental domain for G. However, we eventually want to be able to choose $w = \infty$ for the Fuchsian group we introduce in the next section, and so we need a generalized idea of a Dirichlet polygon that allows for such a choice.

To this end, let $g \in G$ and $\zeta \in \hat{\mathbb{C}}$ such that g is non-trivial and does not fix ζ. Just as with Euclidean isometries, g can be decomposed into two reflections σ_1 and σ_2 in the geodesics L_1 and L_2, respectively. Extend each geodesic into Euclidean circles C_1 and C_2, respectively, where we insist that C_2 contains ζ. Then, we have that $\zeta \in C_2$ by construction and that $\zeta \notin C_1$, since otherwise σ_1 and σ_2 would fix ζ, and hence g would fix ζ. As $\zeta \notin C_1$, there exists a unique half-plane determined by C_1 (either the interior or exterior of C_1) which contains ζ. Label this half-plane H_g. Then, we have the following definition (Definition 9.5.1 in [1]):

Definition 1.2.5. Let G be a Fuchsian group acting on P (any model of the hyperbolic plane) and suppose that $\zeta \in \hat{\mathbb{C}}$ is not fixed by any non-trivial element of G. Then,

$$\Pi_G(\zeta) = \bigcap_{g \in G, g \neq I} H_g.$$

is called the *generalized Dirichlet polygon* with *center* ζ.

With this definition established, we can introduce an important theorem relating the generalized Dirichlet domain of a Fuchsian group and a fundamental domain for the group. We have the following, proven in Beardon (cf. [1], p. 236):

Theorem 1.2.6. *In addition to the assumptions made in Definition 1.2.5, let ζ be an ordinary point of G. Then, $\Pi_G(\zeta)$ is a fundamental domain for G in P. If $\zeta \in P$, then $\Pi_G(\zeta)$ is the Dirichlet polygon $D_G(\zeta)$. If $\zeta = \infty$, then $\Pi_G(\zeta)$ is the region exterior to the isometric circles of all elements of G. Finally, for all h, we have that $h(\Pi_G(\zeta)) = \Pi_{hGh^{-1}}(h\zeta)$.*

The most important statement in this theorem for us concerns the case when $\zeta = \infty$, for in this case we can construct the fundamental domain, called the *Ford fundamental region*, of G using isometric circles.

We conclude this section with a discussion of how certain elements of G interact with its fundamental domain. The above discussion has ideally indicated that each Fuchsian group admits a polygon P as its fundamental domain, called a *convex fundamental polygon*. Each non-trivial element g of G moves P either completely off itself or to a polygon $g(P)$ having one common geodesic with P. A *side* of P is defined to be a set $P \cap g(P)$ which has positive length, and a *vertex* of P is the single point were distinct sides of P intersect (cf. [1], p. 218). Note that only certain elements of G admit a side of P, as only certain elements do not move P entirely off of itself. Let G^* denote the elements of G such that $P \cap g(P)$ is a side of P and let S denote the set of sides s

of P. Now, to each $g \in G^*$ there corresponds an $s \in S$ and each $s \in S$ comes from exactly one g since, if $P \cap g(P) = s = P \cap h(P)$, then $g = h$. Thus, we have a bijective map $\Phi : G^* \mapsto S$ given by $\Phi(g) = P \cap g(P)$. So, there also exists the inverse map $\Phi^{-1} : S \mapsto G^*$ sending the side s to the element g that constructed it. For a given $s \in S$, let g_s denote the image of s under Φ^{-1}. We know that g_s sends s to some $s' \in S$ and then that g_s^{-1} sends this side s' back to s. Thus to each g_s corresponds a pair of sides $\{s, s'\}$ that are paired together under g_s and its inverse. We call each g_s, or simply $g \in G$, a *side-pairing* of P. The usefulness of these side-pairings is exhibited by the final theorem in this section, proven in Beardon (cf. [1], p. 220):

Theorem 1.2.7. *The side-pairing elements G^* of P generate G.*

The last few theorems and some previous discussions provide us with a way to recover some useful algebraic information about G based only on some geometric information about its elements. By Theorem 1.2.6, we can construct a fundamental domain for G using only the isometric circles of its elements. Then, from our discussion on isometric circles, we can find the elements of G that correspond to side-pairings of the fundamental domain. Finally, from the above theorem, we know that these side-pairings generate G. Thus, from the geometric information of isometric circles of elements of G, we can extract the algebraic notion of a generating set for G. This process is used below to extract a generating set for $\Gamma(4)$, which will be useful in future sections.

1.3 Important Facts about $\Gamma(4)$

We devote this section to verifying the many interesting facts about $\Gamma(4)$. The presentation of these facts shall follow a similar order to the presentations of definitions and theorems in the previous section. We shall present each fact as a proposition.

Proposition 1.3.1. $\Gamma(4)$ is a discrete topological group.

Proof: That $\Gamma(4)$ is a topological group follows from the fact that it is a group with a metric (given in §1.1) that induces a topology on it. Let us verify that this is indeed the discrete topology. In order to do this, we wish to show that some singleton set in $\Gamma(4)$ is open, which is equivalent to showing that $\Gamma(4)$ contains an isolated point. We proceed as mentioned in [1] by showing that

$$\inf\{\|A - I\| \mid A \in \Gamma(4), A \neq I\} > 0,$$

which establishes I as an isolated point of $\Gamma(4)$. Note that as we are working with integer entries, the infimum above may be safely exchanged for a minimum. Now, note that as $A \in \Gamma(4)$, $A = \left(\begin{smallmatrix} 4m+1 & 4n \\ 4k & 4l+1 \end{smallmatrix}\right)$ for integers m, n, k, and l. Then, $\|A - I\| = \|\left(\begin{smallmatrix} 4m & 4n \\ 4k & 4l \end{smallmatrix}\right)\| = 4\sqrt{m^2 + n^2 + k^2 + l^2}$. We know that none of m, n, k, or l are 0, as then $A = I$. Thus, $4\sqrt{m^2 + n^2 + k^2 + l^2} \neq 0$. Therefore, the above bound is satisfied and I is an isolated point of $\Gamma(4)$. Hence, $\{I\}$ is open and $\Gamma(4)$ is discrete by Corollary 1.2.1.

Q.E.D.

Proposition 1.3.2. $\mathbb{U}^2$ is invariant under the group action of $\Gamma(4)$.

Proof: Let $A \in \Gamma(4)$. Then, for $z \in \mathbb{U}^2$, $A(z) = \frac{az+b}{cz+d} = \frac{ax+b+iay}{cx+d+icy}$. Rewriting this in standard form, the imaginary part is $\frac{(ad-bc)y}{(cx+d)^2+c^2y^2}$. The denominator is positive, $ad - bc = 1$ as $A \in \Gamma(4)$, and $y > 0$ as $z \in \mathbb{U}^2$. Thus, the imaginary part of $A(z)$ is greater than 0 and so $A(z) \in \mathbb{U}^2$. Therefore, $\mathbb{U}^2$ is invariant under the action by $\Gamma(4)$.

Q.E.D.

Proposition 1.3.3. $\Gamma(4)$ acts discontinuously in $\mathbb{U}^2$

Proof: This follows from the preceding two propositions.

Q.E.D.

Proposition 1.3.4. $\mathbb{U}^2/\Gamma(4)$ is a Riemann surface.

Proof: From the preceding three propositions, $\Gamma(4)$ is a group of Möbius transformations which leaves the the subdomain $\mathbb{U}^2 \subset \hat{\mathbb{C}}$ invariant and which acts discontinuously in $\mathbb{U}^2$. Then, by Theorem 1.2.2, $\mathbb{U}^2/\Gamma(4)$ is a Riemann surface.

Q.E.D.

This result alone does not allow us to say too much about the structure of the Riemann surface formed. However, after considering a few more useful facts about $\Gamma(4)$, we will be able to determine many topological features of this Riemann surface. With this in mind, let us continue.

Proposition 1.3.5. $\Gamma(4)$ is a Fuchsian group.

Proof: This follows from the definition and following discussion of a Fuchsian group given above.

Q.E.D.

Proposition 1.3.6. $\Gamma(4)$ admits a fundamental domain which is a 10 sided ideal polygon in $\mathbb{U}^2$. (see Figure 1)

Proof: By Theorem 1.2.7, $\Gamma(4)$ admits a fundamental domain which is the region exterior of all the isometric circles of its elements. By our earlier discussion regarding isometric circles, this region will be the region between a set of vertical lines that are closest together intersected with the region above the largest isometric circles in $\mathbb{U}^2$. As the vertical lines are distance 4 apart and each isometric circle has diameter $\frac{1}{2}$, there are eight isometric circles between the two vertical lines. Thus, the region in question is above these isometric circles and between the vertical lines, and this is a 10-sided ideal polygon.

Q.E.D.

Note that we may choose any vertical lines distance 4 apart as the sides of our fundamental domain. For our research, we considered the ideal polygon with sides given by $x = -\frac{3}{2}$ and $x = \frac{5}{2}$ and each isometric circle between these two. See Figure 1 below. The elements of $\Gamma(4)$ that maps the vertical lines to one another is given by matrix $a = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and its inverse. From our discussion of isometric circles, the elements that admit the geodesics $[-\frac{1}{2}, 0]$ and $[0, \frac{1}{2}]$ are given by the matrix $b = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and its inverse. To determine the remaining elements of $\Gamma(4)$ that give rise to our fundamental domain, we can observe that each is conjugate to b by translation of some integer distance. Letting $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we find the remaining entries to be:

$$c = t^{-1}bt = \begin{pmatrix} -5 & -4 \\ 4 & 3 \end{pmatrix}, \quad d = tbt^{-1} = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}, \quad e = t^2bt^{-2} = \begin{pmatrix} 9 & -16 \\ 4 & -7 \end{pmatrix},$$

and their inverses. Recall also that these matrices and their inverses constitute the side-pairings of our fundamental domain. Thus, by Theorem 1.2.7, the set $\{a, b, c, d, e\}$ generates $\Gamma(4)$.

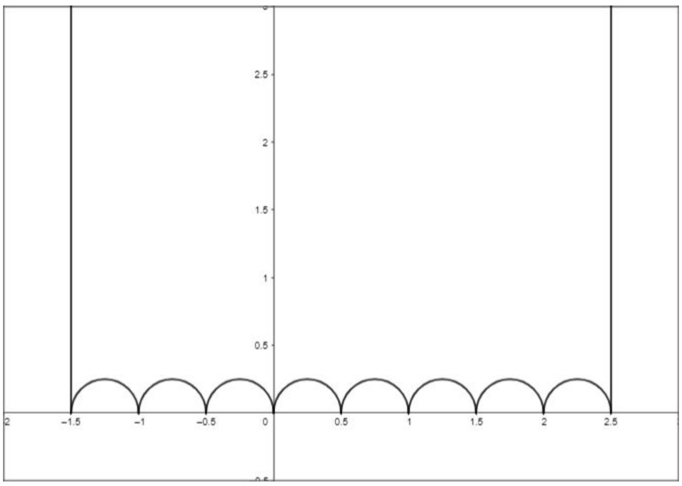

Figure 1: Fundamental Domain for $\Gamma(4)$

Now that we have a better understanding of the structure of the fundamental domain for $\Gamma(4)$, let us conclude this section by determining the structure of the Riemann surface $\mathbb{U}^2/\Gamma(4)$.

Proposition 1.3.7. The Riemann surface $R = \mathbb{U}^2/\Gamma(4)$ is homeomorphic to a sphere with six punctures.

Proof: To form R, we identify, or "glue" each side of the fundamental domain of $\Gamma(4)$ according to the side-pairing elements mentioned above. Each of these elements glues a side s with a side s' where s and s' have a common endpoint; the vertex fixed by the side-pairing. There are five side pairings which fix the five points $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 0)$, and ∞. Thus, the resulting surface has at least five punctures to account for each of the five fixed vertices. Now, observe that the point $(-\frac{3}{2}, 0)$ is glued to $(-\frac{1}{2}, 0)$ by c, which is glued to $(\frac{1}{2}, 0)$ by b, which is glued to $(\frac{3}{2}, 0)$ by d, which is glued to $(\frac{5}{2}, 0)$ by e, which is glued back to $(-\frac{3}{2}, 0)$ by a. Thus, every half-integer is glued to every other half-integer in this construction, and all of them contribute one additional puncture to the surface. Finally, this gluing introduces no genus to the surface. Thus, the resulting Riemann surface is homeomorphic to a sphere with six punctures.

Q.E.D.

2 Cheeger Constant and Important Formulas

In this section we introduce the constant which we are interested in computing for the Riemann surfaces constructed above, the Cheeger Constant. Additionally, we provide some formulas which will be useful in the eventual computation of the Cheeger constant.

2.1 The Cheeger Constant

Here we introduce the Cheeger constant and provide a brief description of what information it tells us about a given surface.

Let M be a Riemann surface. Then, the *Cheeger constant* of M is defined as

$$h(M) = \inf_{E \subset M} \frac{l(E)}{\min\{\text{Area}(A), \text{Area}(B)\}}$$

and is originally introduced in [4]. Here, E is a one dimensional subset of M that divides it into the two disjoint components A and B, and $l(E)$ is the length of E. In practice, we shall take M to be a Riemann surface constructed like that in Theorem 1.2.2 and E to be a curve in $\mathbb{U}^2$ which splits the surface into disjoint pieces. As it will be useful later, let us denote the quantity $\frac{l(E)}{\min\{\text{Area}(A),\text{Area}(B)\}}$ as the *Cheeger ratio* obtained by E. Thus, we can say that the Cheeger constant is the infimum of all Cheeger ratios.

The Cheeger constant tells us some information about the geometry of the surface we are studying. If we have found the Cheeger constant of a given surface, then we have found a curve that is relatively small and, if shrunk down to have 0 length, would separate the surface into disjoint components. The surface itself must pass through this curve and thus this curve creates a sort of "bottleneck" for the surface. Thus we have the description that the Cheeger constant measures the "bottleneckedness" of a surface.

2.2 Formulas Involving Matrices

Here we list a handful of formulas pertaining to the matrices in $\Gamma(4)$, or more generally, $\text{PGL}_2(\mathbb{R})^+$. This will be useful in the eventual computation of the Cheeger constant of our surfaces.

We begin simply with the definition of the trace of a 2x2 matrix. For a given matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the *trace of* A, labelled $\text{tr}(A)$, is defined as $\text{tr}(A) = a+d$. From this definition alone we can derive many useful results. For example, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ be Möbius transformations. We can compute the upper left and lower right entries of AB as $ae+bg$ and $cf+dh$, respectively. Thus, $\text{tr}(AB) = ae+bg+cf+dh$. Similarly, we can compute $\text{tr}(BA) = ea + fc + gb + hd$. As these agree, we have that $\text{tr}(AB) = \text{tr}(BA)$. Thus, trace is invariant under commutation of the involved matrices. Now, let $C = BA$. Then, we have that $\text{tr}(BAB^{-1}) = \text{tr}(CB^{-1}) = \text{tr}(B^{-1}C) = \text{tr}(B^{-1}BA) = \text{tr}(A)$. Thus, trace is also invariant under conjugation by a given matrix (cf. [1], p. 11).

Let us determine the trace of the square of a matrix. Let A be the matrix given above. We compute the upper left and bottom right entries of A^2 as a^2+bc and $bc+d^2$, respectively. Then, we have that $\text{tr}(A^2) = a^2+bc+bc+d^2 = a^2+d^2+2bc+2ad-2ad = a^2+2ad+d^2-2(ad-bc) = (a+d)^2-2(ad-bc) = (\text{tr}(A))^2 - 2\det(A)$. Moreover, if $A \in \text{PSL}_2(\mathbb{Z})$, then $\det(A) = 1$, and hence

$$tr(A^2) = (\text{tr}(A))^2 - 2. \tag{2.2.1}$$

Another useful result regards the fixed points of an isometry. If A, as given above, is a Möbius transformation of $\mathbb{U}^2$, then we can formally define the fixed points of A as those points z such that $z = A(z) = \frac{az+b}{cz+d}$. From this definition, we see that the fixed points satisfy the relation $z = \frac{az+b}{cz+d} \iff cz^2+dz = az+b \iff cz^2+(d-a)z-b = 0$. Then, by the quadratic formula, the fixed points of A are $z = \frac{a-d\pm\sqrt{(d-a)^2+4bc}}{2c} = \frac{a-d\pm\sqrt{a^2+d^2-2ad+4bc}}{2c} = \frac{a-d\pm\sqrt{a^2+2ad+d^2-4ad+4bc}}{2c} = \frac{a-d\pm\sqrt{(\text{tr}(A))^2-4(ad-bc)}}{2c} = \frac{a-d\pm\sqrt{(\text{tr}(A))^2-4}}{2c}$. This result allows us to distinguish between the elliptic, parabolic, and hyperbolic elements of $\Gamma(4)$. If $|\text{tr}(A)| < 2$, then there are two imaginary fixed points of A by the formula, only one of which lies in $\mathbb{U}^2$. Thus, the elliptic elements of $\Gamma(4)$ are those with $|\text{tr}(A)| < 2$. If $|\text{tr}(A)| = 2$, then the formula yields only one real solution. Thus, the parabolic elements of $\Gamma(4)$ are those having $|\text{tr}(A)| = 2$. Lastly, if $|\text{tr}(A)| > 2$, then the formula yields two real solutions. Thus, the hyperbolic elements of $\Gamma(4)$ have $|\text{tr}(A)| > 2$. Also, if $\text{tr}(A)$ is large, then $\sqrt{(\text{tr}(A))^2-4} \approx \text{tr}(A)$. So, the fixed points of A are approximately $\frac{a-d\pm\text{tr}(A)}{2c} = \frac{a-d\pm(a+d)}{2c}$. Taking " + " yields $\frac{a}{c}$ and taking " − " yields $-\frac{d}{c}$.

Next, we have a useful formula that relates the length of a geodesic segment in $\mathbb{U}^2$ with the trace of a certain matrix. To begin, let z_1 and z_2 be arbitrary points in $\mathbb{U}^2$, and the hyperbolic distance between them be labeled l. By appropriate (isometric) conjugations, we may assume that z_1 and z_2 lie on the imaginary axis. Then, we have that

$$l = \cosh^{-1}\left(1 + \frac{(y_2 - y_1)^2}{2y_1y_2}\right),$$

where $(x_2 - x_1)^2$ vanishes as $x_2 = x_1$. Rearranging the argument, we have that $1 + \frac{(y_2-y_1)^2}{2y_1y_2} = \frac{2y_1y_2+y_2^2-2y_1y_2+y_1^2}{2y_1y_2} = \frac{1}{2}(\frac{y_2}{y_1} + (\frac{y_2}{y_1})^{-1})$. Using this fact and taking the hyperbolic cosine of both sides of the above formula yields

$$\cosh(l) = \frac{1}{2}\left(\frac{y_2}{y_1} + \left(\frac{y_2}{y_1}\right)^{-1}\right) = \frac{1}{2}\left(\exp\left(\ln\left(\frac{y_2}{y_1}\right)\right) + \exp\left(-\ln\left(\frac{y_2}{y_1}\right)\right)\right).$$

From this, it follows that $l = \ln\left(\frac{y_2}{y_1}\right)$, or equivalently that $y_2 = \exp(l)y_1$. As $x_2 = x_1 = 0$, we have that $z_2 = \exp(l)z_1$. The matrix that accomplishes the mapping $z_1 \mapsto z_2$ is $\begin{pmatrix} \exp(l) & 0 \\ 0 & 1 \end{pmatrix}$. This matrix does not have determinant 1, but we can rescale by the factor $\exp(-\frac{l}{2})$ to obtain the matrix $A = \begin{pmatrix} \exp(\frac{l}{2}) & 0 \\ 0 & \exp(-\frac{l}{2}) \end{pmatrix}$, which has the desired determinant 1. We than have that $\text{tr}(A) = \exp(\frac{l}{2}) + \exp(-\frac{l}{2}) = 2\cosh\left(\frac{l}{2}\right)$. As the trace is invariant under conjugation, we have the useful formula

$$tr(A) = 2\cosh\left(\frac{l}{2}\right) \tag{2.2.2}$$

for any matrix $A \in \text{PSL}_2(\mathbb{Z})$ moving a point z_1 to z_2 in $\mathbb{U}^2$.

Finally, we have a restriction on the trace of matrices belonging to $\Gamma(4)$. To see this, let $A = \begin{pmatrix} 4m+1 & 4n \\ 4k & 4l+1 \end{pmatrix} \in \Gamma(4)$. Then, we have that $\text{tr}(A) = 4(m + l) + 2$. Moreover, as $\Gamma(4) \subset \text{PSL}_2(\mathbb{Z})$, we have that $(4m+1)(4l+1) - (4k)(4n) = 1 \iff 16(ml - kn) + 4(m+1) = 0 \iff 4(m + l) = 16(kn - ml) \iff m + l = 4(kn - ml)$. Letting $N = kn - ml$, we have that $\text{tr}(A) = 4(m + l) + 2 = 16N + 2$. Lastly, as $2 \equiv -2 \pmod 4$, we have that $\text{tr}(A) = 16N \pm 2$. This result restricts the values of the trace of a matrix in $\Gamma(4)$.

2.3 Formulas from Hyperbolic Geometry

Here we introduce a few useful formulas regarding the area of hyperbolic regions and surfaces.

The first of these formulas is the area of a hyperbolic triangle. If a given hyperbolic triangle in $\mathbb{H}^2$ has angles α, β, and γ, then the area of this triangle is

$$\text{Area} = \pi - (\alpha + \beta + \gamma) \tag{2.3.1}$$

(cf. [1], pg. 150). As a result of this formula, we note that if our triangle is *ideal*, meaning that all of its vertices lie on the boundary of $\mathbb{H}^2$, then each of its interior angles measures 0 and so its area is exactly π.

The second of these formulas regards the area of convex fundamental polygons in $\mathbb{H}^2$. Here, a convex polygon P is one that, given any two points Q_1 and Q_2 in the interior of P, satisfies the convexity condition, meaning that the entire hyberbolic segment Q_1Q_2 also lies in the interior of P. The "fundamental" descriptor from above means that the polygon is the fundamental domain of some Fuchsian group G. We have the following formula, adapted from Corollary 10.4.4 of [1]:

$$\text{h-area}(P) = 2\pi[2(g-1)+t], \tag{2.3.2}$$

where g is the genus of the surface corresponding to P and t is the number of punctures of the surface.

3 Research and Results

In this section we introduce some necessary material to compute the Cheeger constant of a Riemann surface. Next, we compute the Cheeger constant of $R = \mathbb{U}^2/\Gamma(4)$. Finally, we construct a subgroup of $\Gamma(4)$ and its related Riemann surface and compute its Cheeger constant.

3.1 Computing the Cheeger Constant

Here we introduce a procedure and a useful lemma which together allow us to compute the Cheeger constant of a given Riemann surface.

First, we introduce the procedure for computing the Cheeger constant. This is given in [3] and originally adapted from [2]. There are two quantities recorded throughout. The first is H, which represents the current best estimate for the Cheeger constant h and the second is U, which represents the current upper bound on the total length of geodesics that could possibly result in a splitting which reduces H. To *split* a (possibly non-compact) Riemann surface M means to remove a geodesic γ from the surface such that the surface is separated into two disjoint open sets A and B with $\partial A = \partial B = \gamma$. This notation using M, A, and B is used throughout the procedure, which we now state.

(1) First, set $H = 1$ and $U = \text{Area}(M)/2$.

(2) Select a collection $\{\gamma_{i_1}, ..., \gamma_{i_j}\}$ of geodesics which split M into pieces A and B, and which have total length $l(\partial A) = l(\partial B)$ no greater than U.

(3) If $\text{Area}(A) = \text{Area}(B)$, then compute

$$H_0 = h^*(A) = \frac{l(\partial A)}{\text{Area}(A)} = h^*(B) = \frac{l(\partial B)}{\text{Area}(B)},$$

let $s = 0$ and proceed to Step 5.

(4) If $\text{Area}(A) \neq \text{Area}(B)$, without loss of generality let A be the piece of lesser area. Determine the minimum distance d_{i_j} perpendicular from the geodesics into B before the neighborhoods intersect, and minimize the maximum of

$$h^*(A_s) = \frac{l(\partial A)\cosh(s)}{\text{Area}(A) + l(\partial A)\sinh(s)} \text{ and } h^*(B_s) = \frac{l(\partial B)\cosh(s)}{\text{Area}(B) + l(\partial B)\sinh(s)}.$$

Let this minimum be H_0 and record the value of s for which this minimum occurs.

(5) If $H_0 < H$, redefine $H = H_0$ and record the collection $\{\gamma_{i_1}, ..., \gamma_{i_j}\}$. If $H = H_0$, add the collection to the list of collections which achieve H. If $H_0 > H$, do nothing.

(6) If $H \cdot \text{Area}(M)/2 < U$, redefine $U = H \cdot \text{Area}(M)/2$. Otherwise, leave U unchanged.

(7) Return to Step 2 until no further collections of geodesics satisfying the criterion in Step 2 can be found.

In practice, we aim to find a geodesic γ which splits our surface and seems to be a good candidate for attaining the Cheeger constant. That is, it has a relatively small

Cheeger ratio. Then, we argue that no curve of length less than U results in a splitting with a better Cheeger ratio, thus confirming that γ indeed obtains the Cheeger constant. To assist in this argument, we need a way in which to exhaustively list all such curves of lesser length. For this, we have the following lemma adapted from Lemma 2.3 of [3]:

Lemma 3.1.1. *Let Γ be a non-compact, cofinite Fuchsian group with finite-sided fundamental domain $\mathcal{F}$, and let $l > 0$. Choose disjoint Γ-equivariant cusp horoball neighborhoods, and let $\mathcal{F}'$ denote the closure of the complement of these neighborhoods in $\mathcal{F}$. Let $N(\mathcal{F}) = N_{l+\varepsilon}(\mathcal{F})$ denote the closed $(l+\varepsilon)$-neighborhood of $\mathcal{F}'$, and let $G = \{\gamma_0 = Id, \gamma_1, ...\gamma_k\}$ be a set of elements of Γ such that the union of translates $\bigcup_{i=0}^{k} \gamma_i(\mathcal{F})$ covers $N(\mathcal{F}')$. Then any geodesic of length at most l on $\mathbb{U}^2/\Gamma$ must correspond to an element γ_i for some $0 \le i \le k$.*

If we have found a geodesic γ that splits our surface, then we let $l = l(\gamma)$ in the lemma and we thus generate an exhaustive list of geodesics with length less than $l(\gamma)$. These geodesics offer the only possible improvements to the Cheeger ratio attained from γ. For, if we find a geodesic that splits the surface in the same manner as γ but has lesser length, then the Cheeger ratio obtained is smaller.

We conclude this section with a lemma that allows us to consider only certain elements of the list G formed by Lemma 3.1.1 when considering $\Gamma(4)$. Recall that $\Gamma(4)$ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Thus, conjugation by certain elements of $\mathrm{PSL}_2(\mathbb{Z})$ result in elements again belonging to $\Gamma(4)$. In particular, the isometry r accomplishing reflection about the imaginary axis (the line $x = 0$) and the isometry t accomplishing horizontal translation to the right by one unit belong to $\mathrm{PSL}_2(\mathbb{Z})$, so conjugating an element of $\Gamma(4)$ by them returns an element in $\Gamma(4)$. With this in mind, we have the following lemma:

Lemma 3.1.2. *Let $\Gamma(4)$ be the Fuchsian group considered in Lemma 3.1.1 with fundamental domain $\mathcal{F}$ given by the lines $x = -2$ and $x = 2$ and the geodesics connecting each half-integer between them. Let H be the subset of G such that $\bigcup_{i=0}^{n} h_i(\mathcal{F})$ covers the region of $N(\mathcal{F}')$ between the horoballs centered at $(0,0)$ and $(\frac{1}{2}, 0)$. Then, every element* *of G is either in H or conjugate to an element in H by r, t, or some combination of the two.*

Proof: Let G be generated as in Lemma 3.1.1. The subset H consists of the identity, some parabolic elements of G that fix $(0,0)$, some parabolic elements of G that fix $(\frac{1}{2},0)$, and the hyperbolic elements of G whose axis has an endpoint between the horoballs centered at $(0,0)$ and $(\frac{1}{2})$. We form the set H_r by conjugating every element of H by r. By construction, H_r is the group of translates whose union covers the region of $N(\mathcal{F})$ between the horoballs centered at $(-\frac{1}{2},0)$ and $(0,0)$. Thus, $H \cup H_r$ are the translates needed to cover the entire region of $N(\mathcal{F})$ between the horoballs centered at $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$. This set contains the identity, all the parabolic elements of G that fix the origin, some parabolic elements of G that fix $(-\frac{1}{2},0)$, some parabolic elements of G that fix $(\frac{1}{2},0)$, and all the hyperbolic elements of G whose axis has an endpoint between any of the horoball neighborhoods mentioned above.

Now, let $g \in G-(H \cup H_r)$. If g is hyperbolic, then it must have an axis which has an endpoint between some pair of consecutive horoball neighborhoods other than the one mentioned above. Without loss of generality, suppose that the axis of g has an endpoint between the horoballs centered at $(1,0)$ and $(\frac{3}{2},0)$. Then, translating g by one unit to the left, i.e. conjugating by t^{-1} brings the axis of g into alignment with a geodesic having one endpoint between the horoballs centered at the origin and $(\frac{1}{2},0)$. Then, by definition $t^{-1}gt \in H$. If the axis of g had instead been between horoballs such as those centered at $(\frac{3}{2},0)$ and $(2,0)$, then translation by two units to the left and reflection across the y-axis would have brought g into alignment with an element of H. That is, $rt^{-2}gt^2r^{-1}$ would belong to H. As g was chosen to be any hyperbolic element of $G-(H \cup H_r)$, we see that every hyperbolic element of G either belongs to H or is conjugate to it by some combination of r and t. Moreover, as conjugation moves the fixed points of parabolic isometries, an identical argument can be used to show that every parabolic element of $G-(H \cup H_r)$ is conjugate to an element of H. Therefore, every element of G either belongs to H or is conjugate to an element of H by some combination of r and t.

Q.E.D.

While this proof applies to a particular fundamental domain of $\Gamma(4)$, we can adapt to apply to any fundamental domain that has symmetry about a vertical axis, such as the one we consider for most of our work. This lemma allows us to only consider the translates required to cover a region between consecutive horoball neighborhoods, which greatly reduces the number of elements to consider.

3.2 The Cheeger constant of the Surface R

In this section we make a conjecture about the Cheeger constant for the Riemann surface $R = \mathbb{U}^2/\Gamma(4)$. We begin by presenting a geodesic that yields a small Cheeger ratio, verify that this geodesic splits the surface, and run the procedure in §3.1 to produce a reasonable estimate for the Cheeger constant of the surface.

To motivate the selection of a geodesic to serve as our Cheeger constant "candidate", let us make a few observations. The first is that, of all splitting curves of the surface R with a fixed length l, a curve γ produces the smallest Cheeger ratio when R is split into pieces of equal area. This follows from the fact that $\min\{\text{Area}(A), \text{Area}(B)\} \leq \text{Area}(R)/2$, and so

$$\frac{l(\gamma)}{\min\{\text{Area}(A), \text{Area}(B)\}} \geq \frac{l(\gamma)}{\text{Area}(R)/2}.$$

Thus, the Cheeger ratio of γ is smaller if it splits the surface into pieces of equal area. Also, as a result of splitting R into pieces of equal area, γ will also separate the punctures of R evenly, so that each piece A and B has three punctures.

The second observation is that the generating set $\{a, b, c, d, e\}$ of side-pairing elements of $\Gamma(4)$ corresponds to a generating set for the fundamental group of R. Briefly, the *fundamental group* of a topological space X is the set of loops in X, where two loops are considered the same if one can be continually deformed into the other. Now, each of the elements in $\{a, b, c, d, e\}$ correspond to a loop around one of five punctures of R, with the loop around the remaining puncture being formed by the concatenation of the other loops. Thus, words formed from $\{a, b, c, d, e\}$ correspond to the concatenation of generating loops of R, where each generating loop encloses one puncture. As mentioned above, an efficient splitting curve should separate three punctures from the three other punctures, and should thus be a word of length three in $\{a, b, c, d, e\}$. With this in mind, we look for length three words that have fairly small trace, as these correspond to geodesics of fairly short length.

We find the matrix $s = dba^{-1} = \left(\begin{smallmatrix} -11 & 40 \\ -8 & 29 \end{smallmatrix}\right)$ is a length three word with trace 18, which is small considering that the only shorter geodesic length comes from a trace 14 element. The matrix s is hyperbolic and thus has an axis whose endpoints are $(\frac{5\mp\sqrt{5}}{2}, 0)$, which are approximately $(\frac{11}{8}, 0)$ and $(\frac{29}{8}, 0)$. Let γ_s denote the axis of s. To verify that γ_s splits R, we will construct a new fundamental domain for $\Gamma(4)$ which has s as a side-pairing element and such that each remaining side-pairing element pairs sides above γ_s with other sides above the axis of s and similar for sides below γ_s. To this end, let us take the alternate fundamental domain consisting of sides given by $x = 0$ and $x = 4$ together with the geodesics $[0, 1]$, $[1, \frac{4}{3}]$, $[\frac{4}{3}, \frac{3}{2}]$, $[\frac{3}{2}, 2]$, $[2, \frac{5}{2}]$, $[\frac{5}{2}, 3]$, $[3, \frac{7}{2}]$, and $[\frac{7}{2}, 4]$. The side-pairing elements of this fundamental domain, given in terms of the original generators, are $\{a, d, e, aca^{-1}, dba^{-1}\}$. For brevity, let $f = aca^{-1}$. As $c = a^{-1}fa$ and $b = d^{-1}sa$, we see that we can recover the original generators from this new set and thus both sets of elements generate $\Gamma(4)$. So, this new fundamental domain is indeed still a fundamental domain for $\Gamma(4)$.

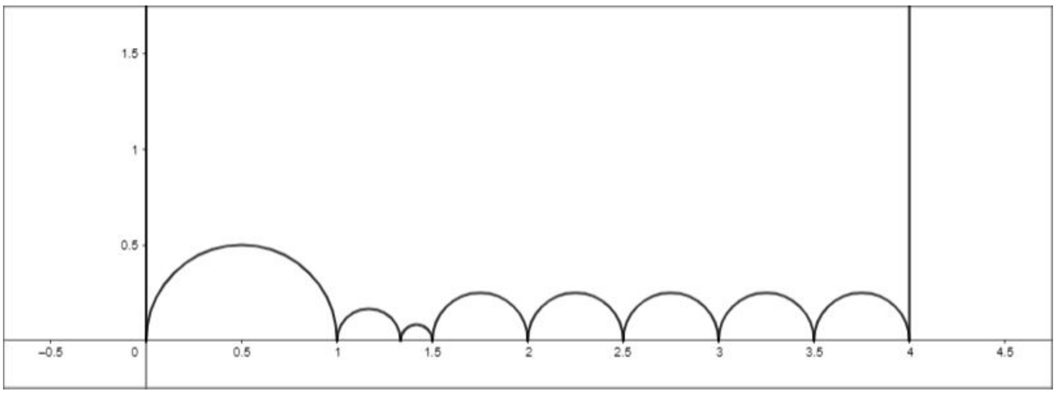

Figure 2: Alternate Fundamental Domain for $\Gamma(4)$

Let us consider which sides are paired together in this fundamental domain. Note that a and its inverse pair the vertical lines together, d and its inverse pair the sides $[0, 1]$ and $[1, \frac{4}{3}]$, e and its inverse pair the sides $[\frac{3}{2}, 2]$ and $[2, \frac{5}{2}]$, f and its inverse pair the sides $[\frac{5}{2}, 3]$ and $[3, \frac{7}{2}]$, and s and its inverse pair the sides $[1, \frac{4}{3}]$ and $[\frac{7}{2}, 4]$ (see Figure 3). Thus, s is a side-pairing element and there are two side-pairings that occur above γ_s, namely those accomplished by a and d, and two side-pairings that occur under γ_s, namely those accomplished by e and f. Thus, when gluing sides together to form R, there are two distinct pieces formed which are separated by γ_s, which has been glued together to form a loop on R. So, γ_s splits R.

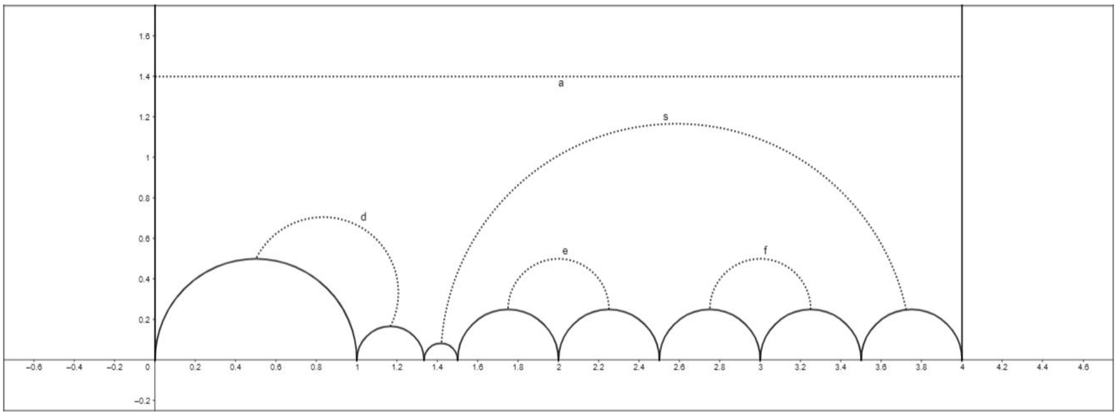

Figure 3: Side-Pairings of Alternate Fundamental Domain for $\Gamma(4)$

Moreover, we claim that γ_s splits R into pieces of equal area. To verify this, let us first determine the area of R. Note that our original fundamental domain (given in Figure 1) can be tiled by eight ideal triangles. As each ideal triangle has area π by formula (2.3.1), the entire fundamental domain has area 8π, and this is the area of R. Now, return to the alternate fundamental domain (given by Figure 2) and consider tiling the section of it above the axis of s with triangles (see Figure 4). Note that there are five triangles in the image. Three of these triangles, T_1, T_2, and T_4 are ideal, and thus the area covered by them is equal to 3π. Triangles T_3 and T_5 have two angles measuring 0 and one nonzero angle. Let α and β denote the nonzero angles of T_3 and T_5, respectively. By construction, α and β are also the angles at which γ_s intersects the fundamental domain. γ_s intersects the fundamental domain at the two geodesics which are paired together by s. Let us consider one of these intersection points. The intersection forms two sets of vertical angles, the sum of which is a straight angle, π. This intersection point is mapped conformally to the other by s. Referring to Figure 4, we see that α is one of these vertical angles and β is the other. So, α and β are supplementary. Thus, the combined area of these triangles is $2\pi - (\alpha + \beta) = 2\pi - \pi = \pi$. Therefore, the entire tiled region has area 4π, which is indeed half the total area of R. Hence, γ_s splits R into pieces of equal area.

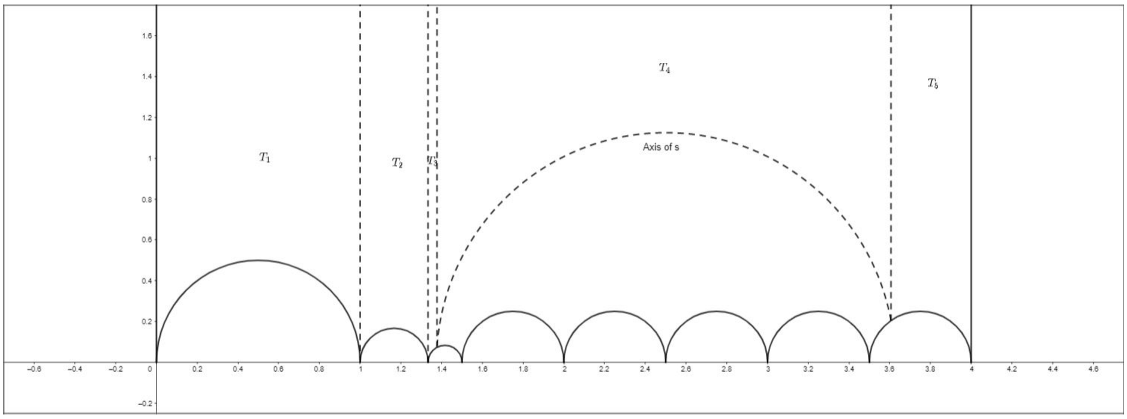

Figure 4: Tiling of Alternate Fundamental Domain for $\Gamma(4)$

Finally, we will use the procedure in §3.1 to produce a reasonable estimate for the Cheeger constant for R. First, we compute the length of γ_s. From §2.2, we have that $\mathrm{tr}(s) = 2\cosh(\frac{l}{2})$, and so $l(\gamma_s) = 2\cosh^{-1}(\frac{18}{2}) \approx 5.77454$. To implement the procedure, we first take $H = 1$ and $U = \frac{\mathrm{Area}(R)}{2} = 4\pi$. We know that γ_s splits R into pieces of equal area, and so by Step 2 we compute $H_0 = \frac{l(\gamma_s)}{\mathrm{Area}(A)} = \frac{2\cosh^{-1}(9)}{4\pi} \approx 0.4595$. As $H_0 < H$, we redefine $H = H_0 \approx 0.4595$ and leave U unchanged. Now, we need only check that no curve of length less than $l(\gamma_s)$ provides a better Cheeger ratio. For this, we would implement Lemma 3.1.2. It should be noted that the actual computation of the Cheeger constant for R has not been completed at present, though current research strongly suggests that it is indeed the value ≈ 0.4595 stated above. Thus, we end with the following conjecture:

Conjecture 3.2.1. The Cheeger constant of the Riemann surface R is $\frac{2\cosh^{-1}(9)}{4\pi} \approx 0.4595$.

3.3 Constructing a Subgroup of $\Gamma(4)$

In this section, we construct an index 2 subgroup of $\Gamma(4)$ and its related fundamental domain. Additionally, we construct the Riemann surface of this fundamental domain, which is a double cover of R.

Recall that $\Gamma(4)$ is generated by the set $\{a, b, c, d, e\}$. From this generating set, we form a new group with generators $\{a^2, ab, ab^{-1}, ac, ac^{-1}, ad, ad^{-1}, ae, ae^{-1}\}$. That is, we take each generator of $\Gamma(4)$ and then inverses and multiply on the left by a, and we take these results as the generators of a new group. Let G denote the group generated by these elements. As each generator is also an element of $\Gamma(4)$, we have that G is a subgroup of $\Gamma(4)$ by construction. Thus, G inherits many of the properties of $\Gamma(4)$. Namely, G acts discontinuously in $\mathbb{U}^2$ and leaves $\mathbb{U}^2$ invariant. Moreover, G is Fuchsian group. As a result, we can form the Riemann surface $R' = \mathbb{U}^2/G$ by gluing sides of the fundamental domain of G together according to its side-pairing elements given by the generating set above.

Let us consider the structure of the fundamental domain of G. Note that the action of multiplying each element of $\{a, b, c, d, e\}$ on the left by a has the effect of combining each side pairing of $\Gamma(4)$ with translation to the right by 4 units. For example, the element b paired the geodesics $[-\frac{1}{2}, 0]$ and $[0, \frac{1}{2}]$, so ab will pair each geodesic with the translate of the other by ± 4 units. That is, $[-\frac{1}{2}, 0]$ is now paired with the geodesic $[4, \frac{9}{2}]$. We can reason similarly for the remaining elements of $\{a^2, ab, ab^{-1}, ac, ac^{-1}, ad, ad^{-1}, ae, ae^{-1}\}$ to determine the structure of the fundamental domain of G. Evidently, we need each of the geodesics of the original fundamental domain together with their translates 4 units to the right, save the geodesic given by $x = 4$, which is absent in this new fundamental domain. Thus, the fundamental domain of G looks like the fundamental domain of $\Gamma(4)$ unioned with its translate by 4 units to the right and having the middle geodesic removed (See Figure 5 below), where the side-pairing elements of G pair a side of the original fundamental domain with one of the sides of the fundamental domain's translate.

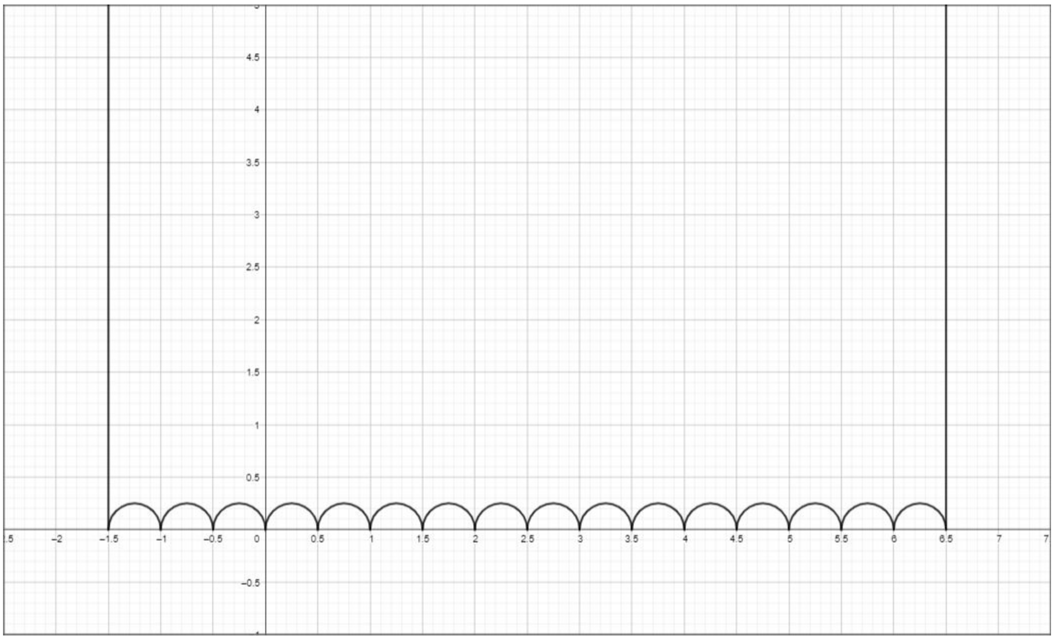

Figure 5: Fundamental Domain for G

To conclude, let us determine the structure of the Riemann surface $R' = \mathbb{U}^2/G$. We proceed in a similar manner as was used to construct R; determining which sets of vertices constitute a puncture of the surface and how the side-pairings determine this. Additionally, we compute the genus of the surface. We formalize the results with the following proposition:

Proposition 3.3.1. The Riemann surface $R' = \mathbb{U}^2/G$ is a genus two surface with six punctures.

Proof: That this surface has six punctures follows from a similar reasoning as above. By the recent example, we see that $(0, 0)$ is glued to $(4, 0)$, and in the same way each of the integer vertices of the original fundamental domain and their translates by 4 units constitute a puncture of the surface. Thus, this surface has at least five punctures. Moreover, it is again the case that each half-integer vertex is glued to every other under the side-pairings of G. Thus, there is one additional puncture of the surface contributed by these vertices.

The fact that this surface has genus two is verified by consideration of formula (2.3.2) introduced in §2.3. As the fundamental domain of G is the union of two copies of the fundamental domain of $\Gamma(4)$, it has area equal to $2(\text{Area}(R)) = 2(8\pi) = 16\pi$. Moreover, it has six punctures by the preceding paragraph. Thus, we may use the formula $\text{Area}(R') = 2\pi[2(g-1)+t]$ to compute the genus of R'. Taking $\text{Area}(R') = 16\pi$ and $t = 6$, we have $16\pi = 2\pi[2(g-1)+6]$, which we can simplify to find that $g = 2$. Thus, R' is a six punctured surface of genus two.

Q.E.D.

3.4 The Cheeger Constant of the surface R'

This section proceeds in a similar manner as §3.2. We first motivate a candidate for the Cheeger constant of R', verify that this element splits the surface efficiently, and then run the procedure to produce a reasonable estimate candidate for the Cheeger constant of the surface.

To motivate a Cheeger constant candidate for R', note the construction that we underwent in order to build the fundamental domain for G. We essentially doubled the fundamental domain for $\Gamma(4)$ and glued it to itself. Thus, a reasonable first guess for a good candidate for the Cheeger constant of R' would be a curve that is double the length of the splitting curve for R. Additionally, any curve having this doubled length should split the surface evenly, separating three punctures from three punctures and one genus from the other. This follows from the discussion at the beginning of §3.2. Recall from §2.2 that, for a matrix A with axis γ, we have the relation $\text{tr}(A) = \exp(\frac{l}{2}) + \exp(-\frac{l}{2})$, where l denotes the length of γ. Let us determine what happens if we double the length of γ, thus replacing l with $2l$. Then, we have that $\exp(\frac{2l}{2}) + \exp(-\frac{2l}{2}) = (\exp(\frac{l}{2}))^2 + (\exp(-\frac{l}{2}))^2 = (\exp(\frac{l}{2}))^2 + (\exp(-\frac{l}{2}))^2 + 2\exp(\frac{l}{2})\exp(-\frac{l}{2}) - 2\exp(\frac{l}{2})\exp(-\frac{l}{2}) = (\exp(\frac{l}{2}) + \exp(-\frac{l}{2}))^2 - 2 = (\text{tr}(A))^2 - 2 = \text{tr}(A^2)$. So, doubling the length of γ equates to squaring the original matrix A. Then, as $s = \begin{pmatrix} -11 & 40 \\ -8 & 29 \end{pmatrix}$ yielded the Cheeger constant of R, $s^2 = s' = \begin{pmatrix} -199 & 720 \\ -144 & 521 \end{pmatrix}$ is a good candidate for attaining the Cheeger constant of R'.

Our first goal is to show that the axis of s', henceforth labeled $\gamma_{s'}$, separates R'. To do this, let us again consider how G was constructed from $\Gamma(4)$. In order to form G, we took the generator a and multiplied it by each generator of $\Gamma(4)$, and their inverses. Let us complete a similar process, this time using the alternate generating set $\{a, d, e, f, s\}$ for $\Gamma(4)$. This yields the side-pairing elements $\{sa, sa^{-1}, sd, sd^{-1}, se, se^{-1}, sf, sf^{-1}, s^2 = s'\}$ for an alternate fundamental domain of G, given by Figures 6 and 7 below. As it is somewhat difficult to determine the sides of this fundamental domain by image alone, we list them explicitly, beginning with the vertical side given by $x = 0$ and progressing to the right to the side given by $x = 4$. The sides of this alternate fundamental domain are as follows: $[0, \infty]$, $[0, 1]$, $[1, \frac{4}{3}]$, $[\frac{4}{3}, \frac{11}{8}]$, $[\frac{11}{8}, \frac{40}{29}]$, $[\frac{40}{29}, \frac{29}{21}]$, $[\frac{29}{21}, \frac{76}{55}]$, $[\frac{76}{55}, \frac{47}{34}]$, $[\frac{47}{34}, \frac{18}{13}]$, $[\frac{18}{13}, \frac{25}{18}]$, $[\frac{25}{18}, \frac{7}{5}]$, $[\frac{7}{5}, \frac{3}{2}]$, $[\frac{3}{2}, 2]$, $[2, \frac{5}{2}]$, $[\frac{5}{2}, 3]$, $[3, \frac{7}{2}]$, $[\frac{7}{2}, 4]$, and $[4, \infty]$.

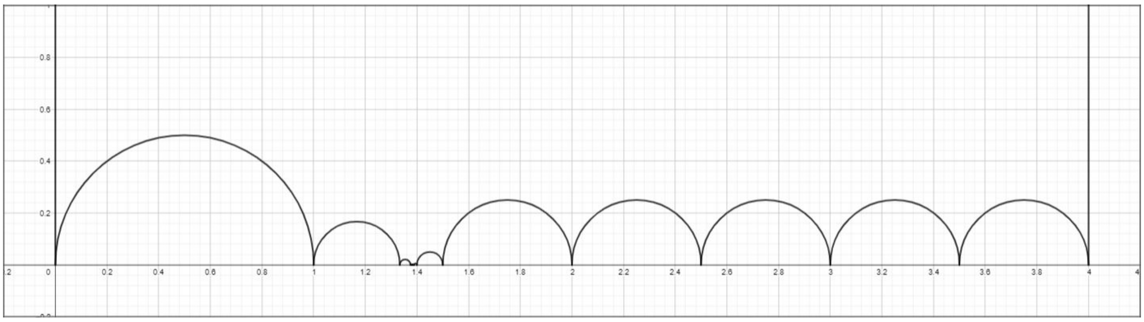

Figure 6: Alternate Fundamental Domain for G

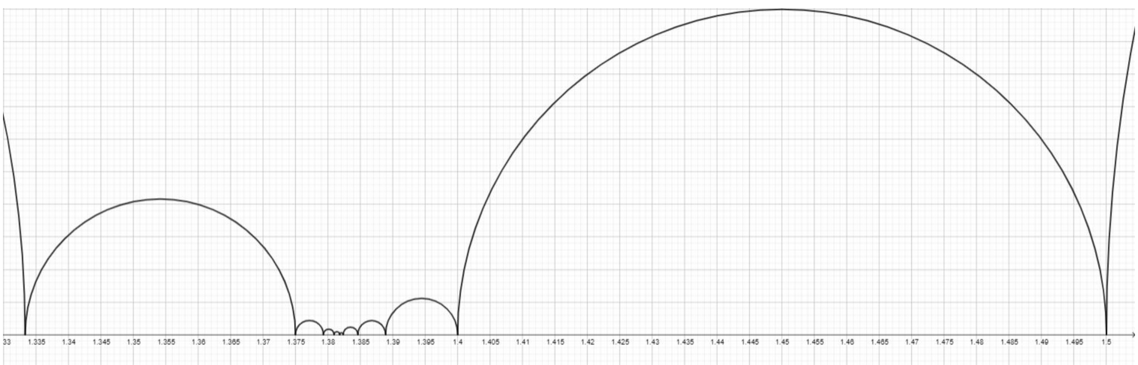

Figure 7: Portion of Alternate Fundamental Domain for G Between $x = \frac{4}{3}$ and $x = \frac{7}{5}$, Enhanced

The geodesic $\gamma_{s'}$ has endpoints at $(\frac{-45\mp\sqrt{405}}{18}, 0)$, which are approximated by $(\frac{199}{144}, 0)$ and $(\frac{521}{144}, 0)$, meaning it intersects the geodesics $[\frac{76}{55}, \frac{47}{34}]$ and $[\frac{7}{2}, 4]$, which are the same geodesics that s' and its inverse pair together. We just need to verify that the remaining side-pairings pair sides above $\gamma_{s'}$ with other sides above $\gamma_{s'}$ and likewise for sides below $\gamma_{s'}$. Observe that sa and its inverse pair the sides $[0, \infty]$ and $[\frac{4}{3}, \frac{11}{8}]$, sa^{-1} and its inverse pair the sides $[4, \infty]$ and $[\frac{11}{8}, \frac{40}{29}]$, sd and its inverse pair the sides $[0, 1]$ and $[\frac{29}{21}, \frac{76}{55}]$, and sd^{-1} and its inverse pair the sides $[1, \frac{4}{3}]$ and $[\frac{40}{29}, \frac{29}{21}]$, which are all side pairings that occur above $\gamma_{s'}$. Meanwhile, se and its inverse pair the sides $[\frac{3}{2}, 2]$ and $[\frac{18}{13}, \frac{25}{28}]$, se^{-1} and its inverse pair the sides $[2, \frac{5}{2}]$ and $[\frac{47}{34}, \frac{18}{13}]$, sf and its inverse pair the sides $[3, \frac{7}{2}]$ and $[\frac{25}{18}, \frac{7}{5}]$, and sf^{-1} and its inverse pair the sides $[\frac{5}{2}, 3]$ and $[\frac{7}{5}, \frac{4}{3}]$, which are all side pairings that occur under $\gamma_{s'}$. Thus, $\gamma_{s'}$ splits the surface R'. Moreover, we can tile the alternate fundamental domain for G in a similar to that done above for the alternate fundamental domain of $\Gamma(4)$, and use a similar argument to find that $\gamma_{s'}$ splits R' into pieces of equal area. Given that the tiling and argument are nearly identical to that done above, we omit them here. Finally, we mention also that the pieces resulting from the splitting by $\gamma_{s'}$ each have one genus. Let A denote one of these pieces. A has area 8π and contains four punctures, three that were present on R' and one that was introduced by the splitting by $\gamma_{s'}$. So, we have that $8\pi = \text{Area}(A) = 2\pi[2(g-1)+4]$, which we can simplify to find that $g = 1$. Thus, A has genus one. Therefore, $\gamma_{s'}$ splits R' into pieces of equal area each having three punctures and genus one.

Finally, we will use the procedure in §3.1 to produce a reasonable estimate for the Cheeger constant of R'. First, we compute the length of $\gamma_{s'}$. By construction, this is double the length of γ_s, and so $l(\gamma_{s'}) = 4\cosh^{-1}(\frac{18}{2}) \approx 11.5491$. To implement the procedure, we first take $H = 1$ and $U = \frac{\text{Area}(R')}{2} = 8\pi$. We know that $\gamma_{s'}$ splits R' into pieces of equal area, and so by Step 2 we compute $H_0 = \frac{l(\gamma_{s'})}{\text{Area}(A)} = \frac{4\cosh^{-1}(9)}{8\pi} \approx 0.4595$. As $H_0 < H$, we redefine $H = H_0 \approx 0.4595$ and leave U unchanged. Now, we need only check that no curve of length less than $l(\gamma_{s'})$ provides a better Cheeger ratio. For this, we would implement a procedure similar to that used in Lemma 3.1.2. As above, it should be noted that the computation for the Cheeger constant of R' has not yet been completed, but current research suggests that it is indeed the value given above. Thus, we end with a similar conjecture as in §3.2:

Conjecture 3.4.1. The Cheeger constant of the Riemann surface R' is $\frac{4\cosh^{-1}(9)}{8\pi} \approx 0.4595$.

That the Cheeger constants of R and R' are likely to be the same is due directly to the related construction of the two surfaces. R' is formed from a fundamental domain whose area is double that of the fundamental domain used to construct R. Moreover, the curve $\gamma_{s'}$ which splits R' is double the length of γ_s, the splitting curve of R. Thus, we have doubled both the length of the splitting curve and the area separated by the splitting curve in moving from R to R', which would leave the Cheeger constant unchanged.

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