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Hidden Symmetries in Classical Mechanics and Related Number Theory Dynamical System

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HIDDEN SYMMETRIES IN CLASSICAL MECHANICS AND RELATED NUMBER THEORY DYNAMICAL SYSTEM

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Abstract

Classical Mechanics consists of three parts: **Newtonian**, **Lagrangian** and **Hamiltonian Mechanics**, where each part is a special extension of the previous part. Each part has explicit symmetries (the explicit Laws of Motion), which, in turn, generate implicit or hidden symmetries (like the Law of Conservation of Energy, etc).

In this Master's Thesis, different types of hidden symmetries are considered; they are reflected in the **Noether Theorem** and the **Poincare Recurrence Theorem** applied to Lagrangian and Hamiltonian Systems respectively.

The Poincare Recurrence Theorem is also applicable to some number theory problems, which can be considered as dynamical systems. In this thesis, we study the problem "The first digits of the powers of integers". A dynamical system interpretation for this problem allows to apply the Poincare Recurrence Theorem to find several unexpected hidden symmetries like rotations on the circle and on the torus, as well as hidden arithmetic progressions in the set of all the exponents.

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Chapter 1

Hidden Symmetries in Classical Mechanics

1.1 Newtonian Mechanics

Newtonian Mechanics studies the motion of a system of point masses in three dimensional Euclidean space. The basic ideas and theorems of Newtonian Mechanics (even when formulated in terms of three dimensional Cartesian Coordinates) are invariant with respect to the 6-dimensional group of Euclidean motions of this space.
A Newtonian potential mechanical system is specified by the masses of the points and by the potential energy. The motions of space which leave the potential energy invariant correspond to the Laws of Conservation.

A. Three Laws of Motion

Newtonian mechanics is the branch of mechanics that is based on Newton's laws of motion. Newton formulated mechanics in terms of three laws of motion. These laws of motion are as following.

1. A body continues in a state of uniform motion unless acted on by a force.

2. A body of mass $\mathbf{m}$ acted on by a force $\mathbf{F}$ accelerates with an acceleration $\mathbf{a}$ such that $\mathbf{F} = \mathbf{ma}$. Indeed, suppose $\mathbf{F} = \mathbf{0}$. Then $\mathbf{ma} = \mathbf{0}$, and since $\mathbf{m} \neq \mathbf{0}$, we get $\mathbf{a} = \mathbf{0}$. Hence $\dfrac{d\boldsymbol{v}}{d\boldsymbol{t}} = \mathbf{0}$, and thus $\boldsymbol{v} = \textbf{constant}$ as it is claimed in the First law.

3. A force $\mathbf{F}$ acting on a body will be accompanied somewhere by an equal and opposite reactive force $\textbf{(-F)}$.

The meaning of the **First Law** is that it establishes the existence of inertial systems of coordinates or inertial frames of references. Once the inertial frames of references are introduced, we can observe that the **First Law** of motion can be derived from the **Second Law**. It is a mere corollary from the **Second Law** in which the force $\mathbf{F}$ is taken to be zero.

The motion of a point particle can be completely described by the particle's position as a function of time in some inertial coordinate system. If the coordinate of a particle's position is expressed by a vector $\mathbf{x(t)}$, then the velocity $\mathbf{v(t)}$ and the acceleration $\mathbf{a(t)}$ are expressed as

$$\boldsymbol{v(t)} = \frac{\boldsymbol{dx}}{\boldsymbol{dt}} \tag{1.1}$$

$$\boldsymbol{a(t)} = \frac{\boldsymbol{d^2x}}{\boldsymbol{dt^2}} \quad . \tag{1.2}$$

In other way, if the coordinate of a particle's position is expressed by a vector $\mathbf{x(t)}$, then the velocity and the acceleration can be expressed in the Newton's notation (the dot on the top):

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{dx}}{\boldsymbol{dt}} = \boldsymbol{v(t)} \tag{1.3}$$

$$\ddot{\boldsymbol{x}} = \frac{\boldsymbol{d^2x}}{\boldsymbol{dt^2}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{dx}}{\boldsymbol{dt}}\right) = \frac{\boldsymbol{dv(t)}}{\boldsymbol{dt}} = \boldsymbol{a(t)} \tag{1.4}$$

Newtonian mechanics is based on application of Newton's Laws of motion which assume that the concepts of distance, time, and mass, are absolute, that is, motion is in an inertial frame of reference. An inertial frame of reference has the property that in this frame when zero net force act upon a body, then that body does not accelerate meaning that body is at rest or is moving at a constant speed with respect to this frame. This fact is represented by the **First Law** of motion, which is a corollary from the **Second Law** of motion as previously mentioned.

B. The Law of Conservation of Energy

The law of conservation energy states that the total energy in an isolated mechanical system is constant, that is the total energy is conserved over the time. We will prove this fact below based on the Newton's Second Law. To prove the law of conservation of energy, we have to introduce the notion of the **Potential Energy**.

Let us consider a particle at initial position x_0 on the x-axis in force field $F(x)$ i.e. the particle whose position is x is subject to the force $F(x)$. The mass-spring system is a prime example. In that case, $F(x) = -kx$, where x is the position of the particle relative to th equilibrium. Intuitively, potential energy of the point mass at x is the work that we must do against the force $F(x)$ to bring the particle to the location x from some given location x_0. It means, we must apply force $-F(s)$ at any point $s, x_0 < s < x$, in order to balance $F(s)$, thus dragging the particle from x_0 to x. This suggests the definition of potential energy as following.

Potential Energy: Potential energy $U(x)$ at x of a point mass in the force field $F(x)$ is defined as the work done from x_0 to x :

$$U(x) = -\int_{x_0}^{x} F(s)ds \tag{1.5}$$

Differentiating both sides and using the Fundamental Theorem of Calculus we get

$$U^{'}(x) = -F(s)|_{s=x} = -F(x) \tag{1.6}$$

Theorem: *The total energy is constant for any motion $x = x(t)$ governed by $F = \ddot{x}$ for a body having unit mass; that is,*

$$\Big(\ddot{x} = F(x)\Big) \iff \Big(\frac{\dot{x}^2}{2} + U(x) = E = \text{const}\Big) \tag{1.7}$$

Proof: $\Longrightarrow$ Let us prove the theorem from forward direction first. So, given $F = \ddot{x}$, we want to prove that

$$\frac{\dot{x}^2}{2} + U(x) = \text{const.}$$

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\dot{x}^2}{2} + U(x)\right] &= \frac{1}{2}\frac{d\dot{x}^2}{dt} + \frac{dU(x)}{dt} \\
&= \frac{1}{2} \cdot 2 \cdot \dot{x} \cdot \frac{d\dot{x}}{dt} + \frac{dU(x)}{dx} \cdot \frac{dx}{dt} \\
&= \dot{x} \cdot \ddot{x} + U^{'}(x) \cdot \dot{x} \\
&= \dot{x}\,\Big(\ddot{x} + U^{'}(x)\Big) \\
&= \dot{x}\,\Big(\ddot{x} - F(x)\Big) \\
&= 0\ .
\end{aligned}$$

Therefore

$$\frac{d}{dt}\left[\frac{\dot{x}^2}{2} + U(x)\right] = 0 \implies \frac{\dot{x}^2}{2} + U(x) = \text{const.}$$

The first part of the theorem is proved.

$\Longleftarrow$ Now we prove the theorem from backward direction: given $\frac{\dot{x}^2}{2} + U(x) = \text{const}$, prove $\ddot{x} = F(x)$. Since $\frac{\dot{x}^2}{2} + U(x) = \text{const}$, differentiating both sides yields:

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\dot{x}^2}{2} + U(x)\right] = 0 \\
\Longrightarrow\ &\frac{1}{2}\frac{d\dot{x}^2}{dt} + \frac{dU(x)}{dt} = 0 \\
\Longrightarrow\ &\frac{1}{2}\cdot 2\cdot \dot{x}\cdot \frac{d\dot{x}}{dt} + \frac{dU(x)}{dx}\cdot\frac{dx}{dt} = 0 \\
\Longrightarrow\ &\dot{x}\cdot\ddot{x} + U^{'}(x)\dot{x} = 0 \\
\Longrightarrow\ &\dot{x}\Big(\ddot{x} + U^{'}(x)\Big) = 0 \qquad [\text{ assume } \dot{x} \neq 0\ ] \\
\Longrightarrow\ &\ddot{x} = -U^{'}(x) = -\frac{d}{dx}\left(-\int_{x_0}^{x} F(s)ds\right) \\
\Longrightarrow\ &\ddot{x} = F(x).
\end{aligned}
$$

Thus, the Theorem is proved completely.

□

1.2 Hidden Symmetry in Lagrangian Mechanics

Lagrangian Mechanics describes motion in a mechanical system by means of the configuration space. The configuration space of a mechanical system has the structure of a differentiable manifold, on which its group of diffeomorphisms acts. The basic ideas and theorems of Lagrangian Mechanics are invariant under this group, even if formulated in terms of local coordinates.

A Lagrangian Mechanical system is given by a configuration space and a function on its tangent bundle, i.e. the Lagrangian Function. Every one-parameter group of diffeomorphisms of a configuration space which fixes the Lagrangian Function defines a conservation of law. A Newtonian potential system is a particular case of Lagrangian system. The configuration space in this case is Euclidean and the Lagrangian function is the difference between the kinetic energy and the potential energy.

A. The Euler-Lagrange Equation

Let us consider that a particle is moving along the x-axis subject to the force $F(x)$ acting in the direction of the line. Let us consider the difference between the potential and kinetic energy of that particle:

$$L(x, \dot{x}) := K - U = \frac{m\dot{x}^2}{2} - U(x). \tag{1.8}$$

The function L depending on x and $\dot{x}$ is called the **Lagrangian**. We will now consider two different cases. In first case, the variables x and $\dot{x}$ are independent i.e. it is not a function of time t. In the second case, x and $\dot{x}$ are functions of time t. Let us differentiate $L(x, \dot{x})$ with respect to x and $\dot{x}$, respectively. Differentiating with respect to x gives:

$$L_x = \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{m\dot{x}^2}{2} - U(x)\right) = 0 - U_x^{'}(x) = -U_x^{'}(x).$$

Differentiating with respect to $\dot{x}$ gives:

$$L_{\dot{x}} = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{m\dot{x}^2}{2} - U(x)\right) = \frac{\partial}{\partial \dot{x}}\left(\frac{m\dot{x}^2}{2}\right) - \frac{\partial}{\partial \dot{x}}\Big(U(x)\Big) = \frac{m}{2}\Big(2\dot{x}\Big) - 0 = m\dot{x}.$$

The term $m\dot{x}$ is called **the momentum of the particle**. Therefore, for the first case we have

$$\begin{cases} L_x = -\ U^{'}(x)\ . \\ L_{\dot{x}} = m\dot{x}\ . \end{cases} \tag{1.9}$$

Now let us consider the second case where both x and $\dot{x}$ are functions of time t.

Differentiating with respect to time t and using equation (1.9) yields

$$\ddot{x} = \frac{d}{dt}\Big(\dot{x}\Big) \iff m\ddot{x} = \frac{d}{dt}\Big(m\dot{x}\Big) = \frac{d}{dt}\Big(L_{\dot{x}}\Big) \tag{1.10}$$

B. The Noether Theorem

The equation (1.9) allows us to extract some hidden symmetry from the dynamical system in question. From Newtonian mechanics we know that $m\ddot{x} = -\,U'(x)$. And from equation (1.9) we know that $L_x = -\,U'(x)$. Therefore, the equation (1.10) can be rewritten as follows:

$$\boxed{\frac{d}{dt}(L_{\dot{x}}) = L_x} \tag{1.11}$$

$$\boxed{\frac{d}{dt}(L_{\dot{x}}) - L_x = 0} \tag{1.12}$$

Each of the equations (1.11) and (1.12) is known as the **Euler-Lagrange Equation**. Namely it leads to the following theorem that expresses a hidden symmetry of the dynamical system.

Noether's Theorem: *Let $x = x(t)$ be a solution of Euler-Lagrange equation. Then*

$$\dot{x}L_{\dot{x}} - L = \text{Constant.}$$

In other words, The Noether's Function $\dot{x}L_{\dot{x}}$ does not change along a particle's trajectory, i.e. it is the first integral of the system.

Proof: If we can prove that $\frac{d}{dt}\Big(\dot{x}L_{\dot{x}} - L\Big) = 0$ then it will follow that $\dot{x}L_{\dot{x}} - L =$

constant. So, we differentiate $\dot{x}L_{\dot{x}} - L$:

$$\begin{aligned}\frac{d}{dt}\Big(\dot{x}L_{\dot{x}} - L\Big) &= \ddot{x}L_{\dot{x}} + \dot{x}\frac{d}{dt}\Big(L_{\dot{x}}\Big) - L_x\dot{x} - L_{\dot{x}}\ddot{x} \\ &= \Big(\ddot{x}L_{\dot{x}} - L_{\dot{x}}\ddot{x}\Big) + \dot{x}\frac{d}{dt}\Big(L_{\dot{x}}\Big) - L_x\dot{x} \\ &= \dot{x}\Big(\frac{d}{dt}L_{\dot{x}} - L_x\Big) \\ &= 0 \qquad \text{[By the Euler Lagrange Equation(1.12)]}\end{aligned}$$

Thus, $\dot{x}L_{\dot{x}} - L$ is constant and we are done.

C. The Equivalence of The Newton's Second Law and Euler-Lagrange Equation

Here, we prove the following fundamental result that ties Newtonian Mechanics and Lagrangian Mechanics.

Theorem: *The Newton's Law $m\ddot{x} = F(x)$ is equivalent to the Euler-Lagrange equation*

$$\frac{d}{dt}(L_{\dot{x}}) = L_x$$

.

Proof: We have just proved that $m\ddot{x} = F(x)$ implies $\frac{d}{dt}\Big(L_{\dot{x}}\Big) = L_x$. We will now prove from backward direction that $\frac{d}{dt}(L_{\dot{x}}) = L_x$ implies $m\ddot{x} = F(x)$.
From equation (1.9)

$$\frac{d}{dt}\Big(L_{\dot{x}}\Big) = \frac{d}{dt}\Big(m\dot{x}\Big) = m\ddot{x}\ . \tag{1.13}$$

Also, from equation (1.6)

$$U^{'}(x) = -F(x)\ . \tag{1.14}$$

So, from equation (1.9) and (1.13) we get

$$L_x = -U^{'}(x) = -(-F(x)) = F(x)\ . \tag{1.15}$$

Therefore, from equation (1.12), (1.14) and the Euler-Lagrange equation we have

$$m\ddot{x} = F(x)\ . \tag{1.16}$$

and the Theorem is thus proved..

1.3 Hidden Symmetry in Hamiltonian Mechanics

Hamiltonian Mechanics is geometry in the phase space. The phase space has the structure of a special manifold which is not described here.

In the previous section we showed how to reformulate the Second Newton's Law in the form of the Euler-Lagrange equation. In this section we will reformulate the Second Newton's Law for a Hamiltonian system.

By definition, the momentum of a particle is given by $p = m\dot{x}$. Let us express the total energy in terms of the momentum p and the position x. Rewrite $p = m\dot{x}$ as $\dot{x} = \dfrac{p}{m}$ (if $m = 1$, then $p = \dot{x}$). Substituting the value of $\dot{x}$ to express the total energy yields

$$H = H(x, \dot{x}) = \frac{m\dot{x}^2}{2} + U(x) \implies H = H(x, p) = \frac{p^2}{2m} + U(x) \tag{1.17}$$

If $m = 1$ then

$$H(x, p) = \frac{p^2}{2} + U(x)$$

The function H, expressed in terms of position the **x** and the momentum **p** is called **the Hamiltonian of the system**. The partial derivatives of the Hamiltonian with respect to x and p yields

$$H_x(x,p) = 0 + U^{'}(x) = -F(x)\ , \tag{1.18}$$

$$H_p(x,p) = 0 + \frac{2p}{2m} = \frac{p}{m} = \dot{x}\ . \tag{1.19}$$

Since $m\ddot{x} = \frac{d}{dt}(m\dot{x}) = \dot{p}$, the Second Newton's Law $m\ddot{x} = F(x)$ can be rewritten in the form $\dot{p} = F(x)$. So, the equations (1.18) and (1.19) can be rewritten as a system of two equations (2n equations in the multidimensional case) which describes the motion in the phase space:

$$\begin{cases} \dot{x} = H_p(x,p)\ , \\ \dot{p} = -H_x(x,p)\ . \end{cases} \tag{1.20}$$

By the Theorem from the **Section C**, this Hamiltonian system is equivalent to the Second Newton's Law. If we consider that the motion of a particle depends on time then we would have $H = \frac{m\dot{x}^2}{2} + U(x,t)$ and $\ddot{x} = -U_x(x,t)$, so the Hamiltonian system thus will be written as

$$\begin{cases} \dot{x} = H_p(x,p,t)\ , \\ \dot{p} = -H_x(x,p,t)\ . \end{cases} \tag{1.21}$$

Therefore we have shown the deep connection between Newtonian, Lagrangian and Hamiltonian mechanics by showing that the Second Newton's Law can be reformulated appropriately into the other two.

So, the vector $\vec{v} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} H_p \\ -H_x \end{pmatrix}$ is attached to each point (x,p) and thus a vector field is obtained. We prove now that the divergency of this vector field is 0.

Indeed:

$$\text{div}(\vec{v}) = \frac{\partial H_P}{\partial x} + \frac{\partial(-H_x)}{\partial p} = \frac{\partial^2 H}{\partial x \partial p} - \frac{\partial^2 H}{\partial p \partial x} = 0.$$

According to the **Liouville's Theorem**, this implies that volumes (areas in 2-dimensional cases) are preserved in time t. **The preservation of volumes is a hidden symmetry in Hamiltonian Systems**.

Another hidden symmetry, even deeper, is the **Poincare Recurrence Theorem** which is a corollary of the Liouville's Theorem. The Poincare Recurrence Theorem will allow us to investigate special dynamical systems on the circle S^1 and on the torus $\mathbb{T}^2$, and tie a special type of Hamiltonian Systems with the Number Theory.

Chapter 2

Continuous and Discrete Dynamical Systems on the Circle $\mathbb{S}^1$ and on the Two Dimensional Torus $\mathbb{T}^2$

2.1 Phase Plane and Phase Space

We know that the second Newton's Law $m\ddot{x} = F(x)$ represents the motion of a particle on a **one dimensional** line. We will now describe a way to think of this system geometrically by increasing the dimension of space. Complete information of a particle's future consists of two pieces of data $(x, \dot{x})$ at some moment of time t. So, instead of two variables: the position on the line x, and the velocity $\dot{x}$, we will consider just one point in the plane with two independent coordinates, x and $\dot{x}$. That is, we consider a two-dimensional point $(x, \dot{x})$, inhabiting in two-dimensional plane. To visualize $(x, \dot{x})$ as a point in a plane, we can treat the velocity $\dot{x}$ as the second dimension. This point $(x, \dot{x})$ is called the **phase point**.Therefore, every point in the phase plane represents a particle located on the line $\mathbb{R}$ at point x and moving with with velocity $\dot{x} = y$ along this line. In multidimensional case, both the points x and $\dot{x}$ have n coordinates, so we have a deal with $n+n = 2n-$dimensional **phase space**.

2.2 Rotation of a Circle $\mathbb{S}^1$

Let us consider the following Newton's equation for a particle moving along a straight line, $x \in \mathbb{R}$:

$$\ddot{x} = -\omega^2 x \tag{2.1}$$

Introduce two new coordinates, x_1 and x_2, as follows: $x_1 = x$ (position) and $x_2 = \dot{x}$ (velocity). The equation of the second order can be rewritten as follows.
Here ω is the angular velocity. Let us consider two points, x_1 and x_2, on a circle. So the point x_1 moves to x_2 at the angular velocity ω. Let $x = x_1$. Therefore $\dot{x_1}$ represents the velocity. But we know from the relationship between the linear velocity and the angular velocity given by $v = \omega r$, where r is the displacement. So $\dot{x_1} = \omega x_2$. Now $\ddot{x} = \ddot{x_1} = \omega \dot{x_2} = -\omega^2 x = -\omega^2 x_1$. Therefore $\omega \dot{x_2} = -\omega^2 x_1$, which implies $\dot{x_2} = -\omega x_1$. So the Second Newton's Law $\ddot{x} = -\omega^2 x$ is equivalent to the system of two linear equations

$$\begin{cases} \dot{x_1} = \omega x_2 \ , \\ \dot{x_2} = -\omega x_1 \ . \end{cases} \tag{2.2}$$

In the phase plane, the vector $\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \omega x_2 \\ -\omega x_1 \end{pmatrix} = \omega \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$ is attached to the point $(x_1, x_2) \in \mathbb{R}^2$, so we get a vector field in $\mathbb{R}^2$. Since the dot product $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet$

$\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = x_1x_2 + x_2(-x_1) = 0$ it follows that the velocity vector $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$ is orthogonal to the positional vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

In order to determine the phase trajectories in the phase plane $\mathbb{R}^2$, we use the Law of Conservation of Energy. We start with the Newtonian's Equation $\ddot{x} = -\omega^2 x$. Then the kinetic energy is

$$K(\dot{x}) = \frac{\dot{x}^2}{2} \ ,$$

and the potential energy is

$$U(x) = -\int (-\omega^2 x) dx = \frac{\omega^2 x^2}{2} \ .$$

Hence the total energy is

$$T = K + U = \frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2} = \text{const} = E \ .$$

But we know that $x = x_1$ and $\dot{x}^2 = \dot{x_1}^2 = \omega^2 x_2^2$. Therefore we can rewrite this as follows:

$$T = \frac{\omega^2 x_2^2}{2} + \frac{\omega^2 x_1^2}{2} \tag{2.3}$$

$$\Longrightarrow \ x_1^2 + x_2^2 = \frac{2E}{\omega^2} = \left(\sqrt{\frac{2E}{\omega}}\right)^2 := r^2 \tag{2.4}$$

We see that equation (2.4) is the equation of a circle centered at origin.Thus the phase trajectories are concentric circles centered at the origin, which implies that the phase flow is the group of rotations around the origin through the angle $\alpha = \omega t$ at any time t.

2.3 Dynamical System on Torus $\mathbb{T}^2$

Consider the system of four linear differential equations in $\mathbb{R}^4$:

$$\begin{cases} \dot{x_1} = \omega_1 x_2 \\ \dot{x_2} = -\ \omega_1 x_1 \\ \dot{x_3} = \omega_2 x_4 \\ \dot{x_4} = -\ \omega_2 x_3 \end{cases} \tag{2.5}$$

It splits into two systems as follows :

$$\begin{cases} \dot{x_1} = \omega_1 x_2 \\ \dot{x_2} = -\ \omega_1 x_1 \quad ; \quad (x_1, x_2) \in \mathbb{R}^2_{1,2} \end{cases} \tag{2.6}$$

and

$$\begin{cases} \dot{x_3} = \omega_2 x_4 \\ \dot{x_4} = -\ \omega_2 x_3 \quad ; \quad (x_3, x_4) \in \mathbb{R}^2_{3,4} \end{cases} \tag{2.7}$$

In each of the two dimensional planes $\mathbb{R}^2_{1,2}$ and $\mathbb{R}^2_{3,4}$, the phase curves are circles, rotating by the angles $\omega_1 t$ and $\omega_2 t$, respectively. The direct product of the two circles is called the **two dimensional torus** $\mathbb{T}^2$:

$$\mathbb{T}^2 = S^1 \times S^1 = \{\ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = C = r_1^2 > 0\ ,\ x_3^2 + x_4^2 = D = r_2^2 > 0\ \}$$

2.4 Rotation of the Circle S^1 and the Torus $S^1 \times S^1$

So, we have two rotations, one on the circle $S^1 = S^1_1$, and the second one on the circle $S^1 = S^1_2$. Together, these rotations can be represented in 3 ways: as a motion of a point along a straight line in $\mathbb{R}^2_{\omega_1,\omega_2}$, as the“jumping trajectory” in the square

$[0, 2\pi) \times [0, 2\pi)$, and as the continuous winding on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. All the three representations are equivalent to each other. Now, we formulate a theorem on a circle rotation which will play the central role in our future reasoning.

Theorem (1): Rotation of Circle

Rotation of circle S^1 by an angle α is:

1. *Periodic, if* $\frac{\alpha}{2\pi} \in \mathbb{Q}$;
2. *Everywhere dense and uniform, if* $\frac{\alpha}{2\pi} \notin \mathbb{Q}$.

Proof: Let us consider the first case where $\frac{\alpha}{2\pi} \in \mathbb{Q}$. Then $\frac{\alpha}{2\pi} = \frac{m}{n}$ for some integer m and n. So, $\alpha n = 2\pi m$. Therefore $\phi + \alpha n = \phi + 2\pi m$, hence the trajectory is periodic.

Now we consider the second case, where $\frac{\alpha}{2\pi} \notin \mathbb{Q}$. Our proof is based on the Pigeon-Hole Principle which states that if $k+1$ *objects lie inside k boxes, then at least one box contains more than one object.* Let us divide S^1 into k equal semi-intervals of length $\frac{2\pi}{k}$ each, and consider $k+1$ points $\{\phi, \phi+\alpha, \phi+2\alpha, \ldots, \phi+k\alpha\} \in S^1$. Then, by the Pigeon-Hole Principle, some two points from this set will be in the same semi interval of the length $\frac{2\pi}{k}$. Let those two points be $\phi + p\alpha$ and $\phi + q\alpha$. If $p > q$, then $\big|(\phi + p\alpha) - (\phi + q\alpha)\big| < \frac{2\pi}{k}$. Let us take $s = p - q > 0$. Then $\big|(\phi + s\alpha) - \phi\big| < \frac{2\pi}{k}$. Now consider the points $\{\phi, \phi+s\alpha, \phi+2s\alpha, \ldots, \phi+Ns\alpha, \ldots\} = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_N, \ldots\}$. Then $\forall n$, $|\phi_{n+1} - \phi_n| < \frac{2\pi}{k}$. Here the property "strictly less than" is due to the irrationality of α.

Let us consider an interval $\triangle \subset S^1$ of arbitrary small length $\varepsilon > 0$ such that $|\triangle| = \varepsilon$ and $\frac{2\pi}{k} < \varepsilon$. Therefore, if k is big enough, $k > \frac{2\pi}{\varepsilon}$, then $\triangle$ contains some $\phi_n = \phi + n\alpha$. Taking ϵ as small as we wish and changing the position of the interval $\triangle$, we get a dense set on the entire circle S^1. Actually, if the number of arcs is finite (but as big as you wish), there are exactly three distinct lengths of arcs on S^1. It implies that for any arc $\triangle \subset S^1$, the number k of $\Big\{\phi_i\Big\}_{i=1}^{N}$ landing to this arc satisfies

$$\lim_{N\to\infty}\frac{k}{N}=\frac{|\triangle|}{|S^1|}=\frac{\epsilon}{2\pi}.$$

By definition, this means that the infinite sequence of angles $\{\phi_0,\phi_1,\phi_2,\ldots,\phi_N,\ldots\}$ is distributed on the circle S^1 **uniformly**. The theorem is proved.

Theorem (2): Rotation on the Torus

If ω_1 and ω_2 are rationally independent numbers, that is $\Big(k_1\omega_1+k_2\omega_2=0\Big) \implies \Big(k_1=k_2=0,\ k_1,k_2\in\mathbb{Z}\Big)$*, then the phase trajectory*

$$\begin{cases} \dot{\phi}_1=\omega_1, \\ \dot{\phi}_2=\omega_2. \end{cases} \tag{2.8}$$

is dense on the torus $\mathbb{T}^2=S^1\times S^1$.

Proof: The solution of the equation (2.8) is

$$\begin{cases} \phi_1(t) &= \phi_1(0)+\omega_1 t, \\ \phi_2(t) &= \phi_2(0)+\omega_2 t. \end{cases} \tag{2.9}$$

If ω_1,ω_2 are rationally **dependent**, i.e. $\Big(k_1\omega_1+k_2\omega_2=0\Big) \implies \Big(k_1=k_2=0,\ k_1,k_2\in\mathbb{Z}\Big)$, then the system

$$\begin{cases} \omega_1 T=2\pi k_2\ , \\ \omega_2 T=-2\pi k_1\ . \end{cases} \tag{2.10}$$

has a solution for T because $\dfrac{\omega_1}{\omega_2}\in\mathbb{Q}$. The solution T gives the sought period of rotation. So, in this case the motion is **periodic**.

But in our case, we have irrational rotation on both circles, which is dense, hence, the direct product will be dense on $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$.

2.5 A Uniform Motion along a Multidimensional Torus

Suppose there are n circles: $S_1^1, S_2^1, S_3^1, \ldots, S_n^1$ and n points $P_1 \in S_1^1, P_2 \in S_2^1, P_3 \in S_3^1, \ldots, P_n \in S_n^1$, which move along these circles with constant angular velocities $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$. Then the angular coordinates change in time t linearly:

$$\begin{cases} \psi_1(t) = \omega_1 t + \phi_1, \\ \psi_2(t) = \omega_2 t + \phi_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_n(t) = \omega_n t + \phi_n, \end{cases} \tag{2.11}$$

where $\phi_1, \phi_2, \phi_3, \ \ldots \ , \phi_n$ are the initial phases. The n-dimensional point $\psi(t) = (\psi_1(t), \psi_2(t), \ \ldots \ \psi_n(t))$ moves along the n-dimensional torus: $\psi(t) \in \mathbb{T}^n = S_1^1 \times S_2^1 \times \cdots \times S_n^1$. The trajectory $\gamma = \{\psi(t)\}$ is either **closed** (i.e. homeomorphic to a circle), or fills a k-dimensional torus $\mathbb{T}^k \subset \mathbb{T}^n, k \leq n$, **everywhere densely**.

The dimension k of the torus $\mathbb{T}^k$ is determined as the maximal number of rationally independent angular velocities $\Big\{\omega_i\Big\}_{i=1}^{k} : \Big(\Gamma_{i_1}\omega_{i_1} + \Gamma_{i_2}\omega_{i_2} + \ \ldots \ + \Gamma_{i_k}\omega_{i_k} = 0\Big) \implies \Big(\Gamma_{i_1} = \Gamma_{i_2} = \ \ldots \ = \Gamma_{i_k} = 0 \ ; \ \Gamma_{i_1}, \Gamma_{i_2}, \ \ldots \ \Gamma_{i_k} \in \mathbb{Z}\Big)$.

For example, if $n = 2$, then by Theorem (2), the phase trajectory either densely fills the whole torus $\mathbb{T}^{n=2}$, or is periodic. This means that the discrete time rotation fills this smooth trajectory densely. But the periodic trajectory on $\mathbb{T}^2$ is a one-dimensional torus $\mathbb{T}^1 = S^1$ which is filled densely by the discrete set of points

$\psi(1), \psi(2), \psi(3), \cdots \in \mathbb{T}^2$.

Thus, the following Theorem holds:

Theorem (3): *The* $2n$ *dimensional system of differential equations:*

$$\begin{cases} \dot{x_1} = \omega_1 x_2 \\ \dot{x_2} = -\ \omega_1 x_1 \\ \dot{x_3} = \omega_2 x_4 \\ \dot{x_4} = -\ \omega_2 x_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1} = \omega_n x_{2n} \\ \dot{x}_{2n} = -\ \omega_{2n-1} x_{2n-2} \end{cases}$$

fills densely either the entire n-dimensional torus $\mathbb{T}^n \subset \mathbb{R}^{2n}$ *or densely some k-dimensional torus* $\mathbb{T}^k \subset \mathbb{T}^n$*, or is a closed winding of* $\mathbb{T}^n$*.*

Chapter 3

The First Digits of 2^n , 3^n , ... as The Dynamical System of Rotation of Circle S^1

3.1 Benford's Law

Benford's law, also called the *Newcomb–Benford law*, the law of anomalous numbers, or the *first-digit law*, is an observation about the frequency distribution of leading digits in many real-life sets of numerical data. Though it is mostly known as Benford's Law, it was first observed by notable astronomer-mathematician Simon Newcomb fifty years before Benford. The law states that the the probability of the first digit d is given by:

$$P(d) = \log_{10}\left(1 + \frac{1}{d}\right) \tag{3.1}$$

We will use this Benford's Law to discuss the probability of a certain digit of being the first digit of 2^n.

3.2 2^n as rotations of a unit circle $\mathbb{S}^1$

Let us consider the first digit, A, of 2^n : $A \in \{1, 2, \ldots, 9\}$.

n	1	2	3	4	5	6	7	8	9	...
2^n	2	4	8	1	3	6	1	2	5	...

Table 3.1: First Digits of 2^n

Generally, A could be a multidigital integer: $A = \overline{a_1 a_2 \ldots a_r}$ where $a_1, a_2 \ldots, a_r$ are the digits. Then, $2^n = A \ldots = \{a_1 a_2 a_3 \ldots a_r \ldots\ldots\}$. Here, $\{\ldots\ldots\}$ represents some other unknown digits of2^n after the digits $\{a_1 a_2 a_3 \ldots a_r\}$. Denote the number of unknown digits as m. This following inequalities are clearly hold:

$$\begin{aligned}
&A000\ldots0 \leq 2^n < (A+1)000\ldots0 \\
&\implies A \cdots 10^m \leq 2^n < (A+1) \cdots 10^m \\
&\implies \log(A) + m \leq n\log(2) < \log(A+1) + m \\
&\implies m \leq -\log(A) + n\log(2) < \Big(\log(A+1) - \log(A)\Big) + m \\
&\implies m \leq -\log(A) + n\log(2) < m + \log\Big(\frac{A+1}{A}\Big) \\
&\implies \Big\{-\log(A) + n\log(2)\Big\} \in \Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big)
\end{aligned}$$

Therefore, the quantity $\Big\{-\log(A) + n\log(2)\Big\}$ will be in the interval $\Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big)$ if and only if A is the first digits of 2^n and m is the number of digits in mantissa. We consider the real number line $\mathbb{R}$. So, $-\log(A)$ will be located below zero. Then we keep adding $\log(2)$ for n times. If, after adding $\log(2)$ for n times, the quantity $\Big\{-\log(A) + n\log(2)\Big\}$ lands in the interval $\Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big)$, we will have our desired "first-digit" number A for the first value of n. Thus, the particle starts from point $x_0 - \log(A)$ on the real line $\mathbb{R}$, and then jumps to the points $x_1 = -\log(A) + \log(2)$, $x_2 = -\log(A) + 2\log(2) = x_1 + \log(2)$, $x_3 =$

$-\log(A) + 3\log(2) = x_2 + \log(2)$, $\ldots$, $x_n = -\log(A) + n\log(2)$. We explain this concept with a particular example as following.

Suppose, we want to find some values of n for which the first digit of 2^n is 7. I have used **Python Programming Language** to find the values of n for this particular case. It shows that, for $n = 46$ we find the first digit 7, i.e. **2^{46} starts with 7**. The value of 2^{46} is $70368744177664 = 7.0368744177664 \times 10^{13}$. So in this case, $n = 46$, $A = 7$, and $m = 13$. Therefore, for these values the quantity $\Big\{ -\log(A) + n\log(2)\Big\}$ will land in the interval $\Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big)$ according to our finding. Now,

$$\Big\{ -\log(A) + n\log(2)\Big\} = -\log(7) + 46\log(2) = 13.00228176.$$

And

$$\Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big) = \Big[13\ , 13 + \log(1 + \frac{1}{7})\Big) = \Big[13\ , 13.05799195\Big).$$

So,

$$13.00228716 \in \Big[13\ , 13.05799195\Big).$$

Now, for example, we want to show that **the first digit of 2^{49} is not 7** by the method in our finding. We can use a programming language to find the value of higher exponents. The value of 2^{49} is $2^{49} = 562949953421312 = 5.62949953421312 \times 10^{14}$. Here, $A = 7$, $n = 49$ and $m = 14$. Now

$$\Big\{ -\log(A) + n\log(2)\Big\} = -\log(7) + 49\log(2) = 13.90537175.$$

And

$$\Big[m\ ,\ m + \log\big(1 + \frac{1}{A})\Big) = \Big[14\ , 14 + \log(1 + \frac{1}{7})\Big) = \Big[14\ , 14.05799195\Big).$$

So,

$$13.90537175 \notin \Big[14\ , 14.05799195\Big).$$

Therefore, the value of 2^{49} will not have 7 as its first digit.

Questions:

1. *Can we find an integer* $m \in \mathbb{Z}^+$ *for which* $x_n \in \Big[m, m+\Delta\Big) = \Big[m, m+\log\big(1+\frac{1}{A})\big)$?

2. *How many pairs* $\Big(n, m\Big)$, $n \in \mathbb{N}$, $m \in \mathbb{N}$ *are there for which* $x_n \in I_m = \Big[m, m+\Delta\Big)$?

3. *What is the probability that* 2^n *starts with the number* A *?*

If $2^n = A\ldots$, then $x_n = -\log(A) + n\log(2)$ belongs to some interval $\Delta_m = \Big[m, m+\log(1+\frac{1}{A})\Big)$, the number of which is infinite, as it will be shown below.
Let us write all the possible n's and for each n we write below it whether 2^n starts with the number A, or it does not. This is the same as to write:

$$x_n = -\log(A) + n\log(2) \in \Delta_m\ ,$$

where Δ_m is the semi-interval of length $\Delta = \log\left(1+\frac{1}{A}\right)$.
Thus, in the list of the integers $\{1, 2, 3, \ldots, n\}$, we mark the numbers which correspond to some m and thus to some semi-interval Δ_m :

1	2	3	4	5	$\cdots$	$\cdots$	$\cdots$	$n-3$	$n-2$	$n-1$	n
-	-	-	m_1	-	m_2	-	m_3	-	m_3	$\ldots$	m_k

So, we have k pairs (n, m) for which $x_n \in \Delta_m$. Then, we have the function $\frac{k}{n}$, the meaning of which is the frequency of appearance x_n in the interval Δ_m, or, in other words, the frequency of appearance those exponents n for which 2^n starts with A.

Now, let $n \to \infty$. Then, if the limit

$$\lim_{n\to\infty} \frac{k}{n} = P$$

exists, we call this limit the *probability of 2^n to start with the number A.* Before answering the question posed, we give an example.

Example: Ask the same questions for the particular number $A = 7$: **How often does $\mathbf{2^n}$ start with digit 7 ?**

Now,

$$-\log(A) = -\log(7) = -0.84509804.$$

We would like to find the intervals

$$\Delta_m = \Big[m, m + \log\big(1 + \frac{1}{7})\Big) = \Big[m, m + 0.058\Big)$$

such that

$$\boxed{-\log(7) + n\log(2) \in \big[m, m + 0.058\big)} \tag{3.2}$$

Calculation shows that, if $n \leq 45$, there is no such an interval Δ_m i.e.

- $x_0 = -\log(7) + 0\cdot\log(2) = -0.845 \notin \big[-1, -1 + 0.058\big) = \big[-1, -0.942\big) = \Delta_{-1}$
- $x_1 = -\log(7) + 1\cdot\log(2) = -0.544 \notin \big[-1, -0.942\big) = \Delta_{-1}$
- $x_2 = -\log(7) + 2\cdot\log(2) = -0.24 \notin \big[0, 0.058\big) = \Delta_0$
- $x_3 = -\log(7) + 3\cdot\log(2) = 0.058 \notin \big[0, 0.058\big) = \Delta_0$

Note that, the interval $\Delta_0 = [0, 0.058)$ i.e. it is an open interval therefore $0.058 \notin \Delta_0$.

Coming back to the inequalities, we have

$$- \log(7) + 3 \cdot \log(2) \notin [\ 0, \log\left(\frac{8}{7}\right)\ ) = \Delta_0$$

$$\implies 7 \cdot 10^0 \leq 2^3 = 8 < 8 \cdot 10^0$$

which is not true. So, despite the fact the making three jumps from $\log(A) = -\log(7)$ with the step $\log(2)$ for three times, jumping point lends at **exactly the right end** of the interval $\Delta_0 = \left[0, \log\left(\frac{8}{7}\right)\right)$.

So, the third jump does not lead to any of the intervals Δ_m, because the only possibility could be Δ_0, but $\log\left(\frac{8}{7}\right) \notin \Delta_0$, because the inequality $-\log\left(7\right) + 3 \cdot \log\left(2\right) < \log\left(1 + \frac{1}{8}\right)$ is strict, and it does not hold.

Likewise,

- $x_4 = -\log(7) + 4 \cdot \log(2) = 0.359 \notin \left[0, 0.058\right) = \Delta_0$ and $\notin \left[1, 1.058\right) = \Delta_1$
- $x_5 = -\log(7) + 5 \cdot \log(2) = 0.66 \notin \left[0, 0.058\right) = \Delta_0$ and $\notin \left[1, 1.058\right) = \Delta_1$
- $x_6 = -\log(7) + 6 \cdot \log(2) = 0.96 \notin \left[0, 0.058\right) = \Delta_0$ and $\notin \left[1, 1.058\right) = \Delta_1$
- $x_7 = -\log(7) + 7 \cdot \log(2) = 1.262 \notin \left[1, 1.058\right) = \Delta_1$ and $\notin \left[2, 2.058\right) = \Delta_2$

  ....

- $x_{45} = -\log(7) + 45 \cdot \log(2) = 12.70125 \notin \left[12, 12.058\right) \implies x_{45} \notin \Delta_m \forall m$

However,

$$x_{46} = -\log\left(7\right) + 46 \cdot \log\left(2\right) = 13.002 \in \left[13, 13.058\right) = \Delta_{13}.$$

This means that $2^{46} = 7\ldots$, i.e. the value 2^{46} starts with the digit 7 for $n = 46$, $m = 13$ and indeed, $2^{46} = 70368744177664$.

Next calculations, where we skip those n's for which $x_n \notin \Delta_m$, show the following

- $-\log(7) + 46 \cdot \log(2) = 13.002 \in \big[13, 13.058\big) = \Delta_{13}$; $\boxed{n = 46\ ,\ m = 13}$

- $-\log(7) + 56 \cdot \log(2) = 16.012 \in \big[16, 16.058\big) = \Delta_{16}$; $\boxed{n = 56\ ,\ m = 16}$

- $-\log(7) + 66 \cdot \log(2) = 19.022 \in \big[19, 19.058\big) = \Delta_{19}$; $\boxed{n = 66\ ,\ m = 19}$

- $-\log(7) + 76 \cdot \log(2) = 22.033 \in \big[22, 22.058\big) = \Delta_{22}$; $\boxed{n = 76\ ,\ m = 22}$

- $-\log(7) + 86 \cdot \log(2) = 25.043 \in \big[25, 25.058\big) = \Delta_{25}$; $\boxed{n = 86\ ,\ m = 25}$

- $-\log(7) + 96 \cdot \log(2) = 28.053 \in \big[28, 28.058\big) = \Delta_{28}$; $\boxed{n = 96\ ,\ m = 28}$

But,

$$-\log\big(7\big) + 106 \cdot \log\big(2\big) = 31.064 \notin \big[31, 31.058\big) = \Delta_{31}.$$

Also for each r , $97 \leq r \geq 148$, calculation shows that

$$-\log\big(7\big) + 106 \cdot \log\big(2\big) \notin \Delta_m \text{ for any value of } m\ .$$

So far, we have the following table :

n	$1, 2, \cdots, 45$	46	56	66	76	86	96	97 , 98 , $\cdots\cdots$, 148
m	no such m's	13	16	19	22	25	28	no such m's

Table 3.2: The values of n (up to 148) and m for which 2^n starts with the digit 7

We can calculate the frequency of appearance of the pairs (n, m) for which $x_n \in \Delta_m$:

$$\lambda_n = \frac{\#\{m\}}{n} = \begin{cases} 0 \ , \ \text{if} \ \ n = 1, 2, 3, \cdots, 45 \ ; \\ \frac{1}{46} \ , \ \text{if} \ \ n = 46 \ , \ \lambda_{46} = 0.0217 \ ; \\ \frac{2}{56} \ , \ \text{if} \ \ n = 56 \ , \ \lambda_{56} = 0.0357 \ ; \\ \frac{3}{66} \ , \ \text{if} \ \ n = 66 \ , \ \lambda_{66} = 0.0454 \ ; \\ \frac{4}{76} \ , \ \text{if} \ \ n = 76 \ , \ \lambda_{76} = 0.0526 \ ; \\ \frac{5}{86} \ , \ \text{if} \ \ n = 86 \ , \ \lambda_{86} = 0.0581 \ ; \\ \frac{6}{96} \ , \ \text{if} \ \ n = 96 \ , \ \lambda_{96} = 0.0625 \ ; \end{cases}$$

And we can calculate the frequencies for all n's :

n	$1, 2, \cdots, 45$	46	47	48	$\cdots$	55	56	57	$\cdots$	65	66	$\cdots$
λ_n	0	$\frac{1}{46}$	$\frac{1}{47}$	$\frac{1}{48}$	$\cdots$	$\frac{1}{55}$	$\frac{2}{56}$	$\frac{2}{57}$	$\cdots$	$\frac{2}{65}$	$\frac{3}{66}$	$\cdots$

Table 3.3: The frequency of λ_n for the value of n's

Next calculations show the following results:

- $-\log(7) + 149 \cdot \log(2) = 44.00837 \in \big[44, 44.058\big) = \Delta_{44}$; $\boxed{n = 149 \ , \ m = 44}$
- $-\log(7) + 159 \cdot \log(2) = 47.01867 \in \big[47, 47.058\big) = \Delta_{47}$; $\boxed{n = 159 \ , \ m = 47}$
- $-\log(7) + 169 \cdot \log(2) = 50.02897 \in \big[50, 50.058\big) = \Delta_{50}$; $\boxed{n = 169 \ , \ m = 50}$
- $-\log(7) + 179 \cdot \log(2) = 53.03927 \in \big[53, 53.058\big) = \Delta_{53}$; $\boxed{n = 179 \ , \ m = 53}$
- $-\log(7) + 189 \cdot \log(2) = 56.04957 \in \big[56, 56.058\big) = \Delta_{56}$; $\boxed{n = 189 \ , \ m = 56}$

But,

$$-\log\big(7\big) + 199 \cdot \log\big(2\big) = 59.0598 \notin \big[59, 59.058\big) = \Delta_{59}.$$

Hence, the first digit of 2^{199} is not 7. And indeed, $2^{199} = 8.03469022129 \times 10^{59}$.

We can now calculate the new frequencies as following :

n	149	159	169	179	189	$190, \cdots, 241$
m	44	47	50	53	56	no such m's

Table 3.4: The values of n (up to 241) and m for which 2^n starts with the digit 7

And the frequency distributions are as following :

$$\lambda_n = \frac{\#\{m\}}{n} = \begin{cases} \dfrac{7}{149}\ , \ \text{if} \ \ n = 149\ ,\ \lambda_{149} = 0.0469\ ; \\ \dfrac{8}{159}\ , \ \text{if} \ \ n = 159\ ,\ \lambda_{159} = 0.0503\ ; \\ \dfrac{9}{169}\ , \ \text{if} \ \ n = 169\ ,\ \lambda_{169} = 0.0526\ ; \\ \dfrac{10}{179}\ , \ \text{if} \ \ n = 179\ ,\ \lambda_{179} = 0.0558\ ; \\ \dfrac{11}{189}\ , \ \text{if} \ \ n = 189\ ,\ \lambda_{189} = 0.0582\ ; \end{cases}$$

n	149	159	169	179	189
λ_n	0.0469	0.0503	0.0532	0.0558	0.0582

Table 3.5: The frequency of λ_n for the value of n's from 149 to 189

Therefore, we see that the frequency oscillates, but from time to time it becomes close to $\log\left(\frac{8}{7}\right) = 0.05799\cdots \approx 0.058$ for $n = 96$ and $n = 189$.

3.3 Reduction of jumping along the line to the rotation of the unit circle S^1

In this subsection, we answer to all the questions (1), (2), and (3) posed in the previous section.

Theorem (4):

1. *Let,* $x_n = -\log(A) + n\log(2)$*. Then, there exists an integer* m *such that* $x_n \in \big[m, m + \log\big(1 + \frac{1}{A}\big)\big) = \Delta_m$*.*

2. *There are infinitely many pairs of integers* (n, m) *such that* $x_n \in \Delta_m$*.*

3. *The probability that* 2^n *starts with the number* A *exists, i.e. there exists the limit* $P_A = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \Big(\frac{\#m}{n}\Big)$ *, where* $2^n = A\ldots$*. This probability equals* $P_A = \log\big(1 + \frac{1}{A}\big)$*.*

Proof : We prove (1) and (2) simultaneously.First, we reduce the motion (jumping) along the line $\mathbb{Z}$ to the rotation of the unit circle S^1, as follows:

(1) We shift all the unit semi-interval $I_m = \big[m, m{+}1\big) \supset \Delta_m = \big[m, m{+}\log\big(1{+}\frac{1}{A}\big)\big)$ by the integer m to the left to the semi-interval $I_0 = \big[0, 1\big)$. Then, all the semi-intervals Δ_m , $m = 0, \pm1, \pm2, \pm3, \ldots$ will coincide with the semi-interval $\Delta_0 = \big[0\, , \log\big(1{+}\frac{1}{A}\big)\big)$. As a result, we obtain infinitely many coincided unit semi-intervals $I_0 = \big[0, 1\big)$ and infinitely many coincided semi-interval Δ_0 of length $\varepsilon_A = \log\big(1{+}\frac{1}{A}\big)$ inside the semi-interval $I_0 = \big[0, 1\big)$.

Note that, if $x_n = -\log(A) + n\log(2) \in I_m$, then $\{x_n\} = x_n - [x_n] \in I_0$ because $[x_n] = m$ since $x_n \in I_m$ and $0 < \{x_n\} < 1$.

Note 1: $\{x_n\} \neq 0$, $\{x_n\} \neq 1$ because $\log(2) \notin \mathbb{Q}$.

Note 2: $\{x_n\}$ is called the *fractional part* of x_n which is also called **the mantissa**. For example, if $A = 7$, then $\{x_{76}\} = \{-\log(7) + 76\log(2)\} = \{22.033181\ldots\} = 0.33181\ldots \in I_0$ and moreover $\{x_{76}\} \in \Delta_0$.

Also, $\{x_n\}$ is denoted as $\{x_n\} = x_n(\text{mod } 1)$, because $\{x_n\} \in [0, 1)$ and obtained from x_n by shift. Once we jump along the straight line, the operation $x_n \longmapsto x_n(\text{mod } 1) = \{x_n\}$ reduces our jumping along the circle S^1 of length 1. To do that, we have to identify the end $x = 1 \in I_0 = [0, 1)$ with the end $0 \in I_0$.

And the jump $n\log(2) \in \mathbb{R}$ will correspond to n jumps, or rotations, by angle

$$\alpha = \log(2).$$

Since $\log(2) \notin \mathbb{Q}$, by Theorem (1), we get infinitely many distinct points $-\log(A) + \log(2)$, $-\log(A) + 2\log(2)$, $\log(A) + 3\log(2)$, $\ldots$, $-\log(A) + n\log(2)$, $\ldots$ on the circle S^1 which cover the circle S^1 **everywhere densely and uniformly**. This proves part (1) and (2).

This also means that the frequencies λ_n tends, as $n \to \infty$, to the ratio $\dfrac{|\Delta_0|}{|S^1|} = \dfrac{\epsilon_A}{1} = \epsilon_A = \log\left(1+\dfrac{1}{A}\right)$. Thus, t $\lim_{n\to\infty} \lambda_n$ exists and equals the length of the semi-interval Δ_0, i.e.

$$\lim_{n\to\infty} \lambda_n = \varepsilon_A = \log\left(1 + \frac{1}{A}\right).$$

The last formula proves part (3) and the Theorem (4) is proved.

3.4 Some corollaries from Theorem(4)

(A) Theorem (4) holds not only for the powers of 2, but also for all powers of integers except for powers of 10. Thus 3^n , 4^n , 5^n , $\ldots$ can start with a given integer A infinitely many times, and, moreover, with the probability $\epsilon_A = \log\left(1 + \dfrac{1}{A}\right)$. But, of course, the lists of the exponents n for which these numbers start with the integer A, are distinct. Nevertheless, all the respective frequencies λ_n tend to the same value $\log\left(1 + \dfrac{1}{A}\right)$ which is the length of the interval Δ_0 on the unit circle S^1.

(B) Theorem (4) also proves mathematically the Benford's Law for the problem "The first digits of 2^n , 3^n , 4^n , $\ldots$". Namely, the frequencies or probabilities of the first digits $A = 1, 2, \cdots, 9$ are the following

A	1	2	3	4	5	6	7	8	9
λ_A	$\log(2)$	$\log\left(\frac{3}{2}\right)$	$\log\left(\frac{4}{3}\right)$	$\log\left(\frac{5}{4}\right)$	$\log\left(\frac{6}{5}\right)$	$\log\left(\frac{7}{6}\right)$	$\log\left(\frac{8}{7}\right)$	$\log\left(\frac{9}{8}\right)$	$\log\left(\frac{10}{9}\right)$

Table 3.6: Frequencies of the first digit $A = 1, 2, \cdots, 9$

This also shows, in particular, that 2^n starts with the digit 7 more often than with the digit 8, although 8 shows up many times for small n's, where as 7 appears first time only for $n = 46$.

(C) Theorem (1) and Theorem (4) are both corollaries of the Poincare Recurrence Theorem as follows.

Poincare Recurrence Theorem: *Let $D \subset \mathbb{R}^n$ be a region of a finite volume and $\phi : D \to D$ be a one-to-one transformation that preserves volumes. Then for any point $y \in D$ and for any neighborhood $U = U_y$ of point y there exists point $x \in U_y$ such that $\phi^n(x)$ returns to the region U infinitely many times.*

Proof: Indeed, some of the region $U, \phi(U), \phi^2(U), \cdots$ must intersect, because otherwise the sums of their volumes would be infinite, but volume of $D < \infty$. So, if $\phi^k(U) \cap \phi^m(U) \neq \emptyset$, we have $= \phi^{-k}(\phi^k(U) \cap \phi^m(U)) = \phi^0(U) \cap \phi^{m-k}(U) \neq \emptyset$, but $\phi^0(U) = U$. So $U \cap \phi^{m-k}(U) \neq \emptyset$. This means that point $x \in U \cap \phi^{m-k}(U)$ returns to U. Since U can be arbitrary, we get infinitely many returns to U.

Taking $D = S^1$, we can apply the Poincare Recurrence Theorem to the rotation of the circle S^1. And the rotation of S^1 is equivalent to finding the exponents $\{n\}$ for 2^n to start with a given integer A.

This connection is a **hidden symmetry** between **classical mechanics** and the **number theory**. Indeed, a Hamiltonian dynamical system has $\mathrm{div}(\vec{v}) = 0$, which means, by Liouvilles's Theorem, preserving volumes in the phase space.

And preserving volumes allows us to apply the Poincare Recurrence Theorem to Hamiltonian systems. Changing the exponents 2^n, etc to the logarithmic world, $\{n \log(2)\}$, etc is the same as considering the phase space in number theory.

Chapter 4

Joint Distribution of the first digits of the pair $(2^n, 3^n)$

Let us pose the following problem:

Let A and B are two positive integers. Does there exist a positive integer n such that 2^n starts with the number A and 3^n starts with the number B? If yes, is there infinitely many such n or finitely many? What is the probability that $2^n = A\ldots$ and $3^n = B\ldots$?

To answer this question, we go to the "logarithmic world" by taking decimal logarithms:

$$\begin{cases} 2^n &= A\underbrace{\ldots}_{\text{p}} \\ 3^n &= B\underbrace{\ldots}_{\text{q}} \end{cases} \tag{4.1}$$

where both p and q are unknown. Note that the number $1, \log(2), \text{and } \log(3)$ are rationally independent because for each integers m,n, and k, $2^m \cdots 3^n \neq 10^k$. Then, by Theorem(2) on the motion along the torus$\mathbb{T}^2$, the distance motion of the initial point $(\{-\log(A)\}, \{-\log(B)\})$ along $\mathbb{T}^2$ in the direction $(\log(2), \log(3))$ fills the torus $\mathbb{T}^2$

everywhere densely and **uniformly**. Hence, the following theorem can be proven the same way as Theorem (4) in Section 3.

Theorem 5: *There are infinitely many exponents n for which*

$$2^n = A\ldots \;\; \textit{and} \; 3^n = B\ldots.$$

The probability of such n's equals

$$prob_{A,B} = prob(2^n = a\ldots, 3^n = B\ldots) = \log(1+\frac{1}{A})\cdot\log(1+\frac{1}{B}).$$

Now we solve the same problem for the pair 2^n and 5^n. The answer in this case will be absolutely different than that in the case of 2^n and 3^n. The matter is that 1,log(2), and log(5) are rationally dependent: $2 \cdot 5 = 10$, hence,

$$\log(2) + \log(5) = 1 \implies \log(5) = 1 - \log(2).$$

So, the points

$$Z_n = (x_n, y_n) = \Big(\{n\log(2)\}, \{n\log(5)\}\Big).$$

are located on the diagonal $y = 1 - x$ of the unit square $[0,1)_x \times [0,1)_y$:

$$\begin{cases} x &= n\log(2) \\ y &= n\log(5) \end{cases} \implies x + y = n \underbrace{\implies}_{\text{mod } 1} y = 1 - x \tag{4.2}$$

This observation tells us that if $2^n = A\ldots$ and $5^n = B\ldots$, then B is not arbitrary but is determined by the "logarithmic square-diagram", which is the square $[0,1)\times[0\times 1)$ in the double logarithmic scale. It also allows to determine the first k digits of 2^n and

5^n if they are the same in both numbers by the following theorem:

Theorem 6:

(a) If

$$\begin{cases} 2^n & = \overline{abc\ldots} \\ 5^n & = \overline{abc\ldots} \end{cases} \tag{4.3}$$

then

$$A_k = \overline{a_1 a_2 \ldots a_k} = [\sqrt{10} \cdot 10^{k-1}].$$

We omit a proof of this theorem here.

Regarding arithmetic progression, Gregory Galperin proved the following theorem:

Theorem 7:

(1) For each digit $a \in \{1, 2, \ldots, 9\}$, the set of n's for which 2_n starts with a, is the infinite union of arithmetic progressions of finite lengths depending on the digit a, with almost constant gaps between arithmetic progressions.

(2) If $2^n = 7$, then $\approx 63\%$ of arithmetic progressions have length $l_1 = 5$ and $\approx 37\%$ of arithmetic progressions have length $l_2 = 6$. There are no other arithmetic progressions with the difference $d = 10$. All the gaps between arithmetic progressions have length 53.

(3) The same statement as (2) holds for all other digits, with $d = 10$, but the gaps could be different, and their lengths form a finite short list (depending on the digit a).

(4) The joint distribution for 2^n and 5^n for $\{n | 2^n = 316\ldots, 5^n = 316\ldots\}$ consists of arithmetic progressions with the difference $d = 2136$, and these arithmetic progressions have lengths only 8 and 9 (though the first arithmetic progression of length 4 starts in the middle of arithmetic progression of $l = 8$).

The first arithmetic progression of length $l = 4$ is

$$\boxed{n = 1068, 3204, 5340, 7476\ ;\ d = 2316}$$

Then the next arithmetic progression has gap length $h = 485$ and starts with the third term 5340:

$$\boxed{h = 5825, 7961, 10097, 12233, 14369, 16505, 18641, 20777, 22913\ ;\ d = 2136\ ;\ l = 9}$$

The third arithmetic progression starts 485 after the 7th term 18461;$d = 2136$. In this joint distribution, the arithmetic progressions overlap and have much more difficult structure.

Appendix A

Programming Language

We have used the Python programming language to carry out some results for our thesis.

A.1 The power of 2^n for any power

This coding will provide us the value of 2^n for any power n. To get the result of 2^n for any digit n is as following:

```
print("Hello!!! Welcome to the counting of exponent of 2")
print("Please choose the value of exponent n for the power of 2")
n=input()
print("The value is "+str(2**int(n)))
```

This programming code will prompt you to enter any value of n to calculate the value of 2^n. After any value of n is given as input, it will provide the result. In our thesis, we also needed to know the first digit of the value of 2^n. For that case the coding will have one more command as following:

```
print("Hello!!! Welcome to the counting of exponent of 2")
print("Please choose the value of exponent n for the power of 2")
n=input()
print("The value is "+str(2**int(n)))
print("The left most digit is: "+str(number)[:1])
```

This command will give both the value of 2^n and the first digit of that value.

A.2 The power of 2^n for some range of n

To get the result of 2^n for some range of values of n is as following:

```
for n in range(1,10000):
    number=int(2**int(n))
    if str(number)[:1]=="7":
        print("The values of n are :"+str(int(n)))
```

This programming code will allow us to get the value of 2^n for $n = 1$ to $n = 10000$. If we want to get more values i.e. for higher exponent of n, then we will have to change the value of n in the range command. For example, if we want to find the value of $n = 45$ to $n = 5000$, then range will be $(45, 5000)$.

A.3 The value of n for which 2^n has the same first digit

To calculate the probability of a particular first digit of 2^n in Chapter (3), we used programming language Python. We used it to get all the values of n for which 2^n starts with the digit 7. The command is as following:

```
for n in range(100):
    number=int(2**int(n))
    if str(number)[:1] == "7":
        print("The value is "+str(2**int(n)))
        print("The left most digit is: "+str(number)[:1])
```

These commands will carry the value of exponent of any integers. Note that, caution should be taken when writing the **for** command and **if** command. There should be an **indent** after **for** and **if** command which can be made by the **tab** key.

A.4 The values of n for which 2^n has first digit 7

If we run the code in A.3 for $n = 1$ to $n = 3000$, we will get the following results as exactly displayed in Python:

For the value of n = 46

The left most digit is: 7

For the value of n = 56

The left most digit is: 7

For the value of n = 66

The left most digit is: 7

For the value of n = 76

The left most digit is: 7

For the value of n = 86

The left most digit is: 7

For the value of n = 96

The left most digit is: 7

For the value of n = 149

The left most digit is: 7

For the value of n = 159

The left most digit is: 7

For the value of n = 169

The left most digit is: 7

For the value of n = 179

The left most digit is: 7

For the value of n = 189

The left most digit is: 7

For the value of n = 242

The left most digit is: 7

For the value of n = 252

The left most digit is: 7

For the value of n = 262

The left most digit is: 7

For the value of n = 272

The left most digit is: 7

For the value of n = 282

The left most digit is: 7

For the value of n = 292

The left most digit is: 7

For the value of n = 345

The left most digit is: 7

For the value of n = 355

The left most digit is: 7

For the value of n = 365

The left most digit is: 7

For the value of n = 375

The left most digit is: 7

For the value of n = 385

The left most digit is: 7

For the value of n = 438

The left most digit is: 7

For the value of n = 448

The left most digit is: 7

For the value of n = 458

The left most digit is: 7

For the value of n = 468

The left most digit is: 7

For the value of n = 478

The left most digit is: 7

For the value of n = 488

The left most digit is: 7

For the value of n = 531

The left most digit is: 7

For the value of n = 541

The left most digit is: 7

For the value of n = 551

The left most digit is: 7

For the value of n = 561

The left most digit is: 7

For the value of n = 571

The left most digit is: 7

For the value of n = 581

The left most digit is: 7

For the value of n = 634

The left most digit is: 7

For the value of n = 644

The left most digit is: 7

For the value of n = 654

The left most digit is: 7

For the value of n = 664

The left most digit is: 7

For the value of n = 674

The left most digit is: 7

For the value of n = 727

The left most digit is: 7

For the value of n = 737

The left most digit is: 7

For the value of n = 747

The left most digit is: 7

For the value of n = 757

The left most digit is: 7

For the value of n = 767

The left most digit is: 7

For the value of n = 777

The left most digit is: 7

For the value of n = 830

The left most digit is: 7

For the value of n = 840

The left most digit is: 7

For the value of n = 850

The left most digit is: 7

For the value of n = 860

The left most digit is: 7

For the value of n = 870

The left most digit is: 7

For the value of n = 923

The left most digit is: 7

For the value of n = 933

The left most digit is: 7

For the value of n = 943

The left most digit is: 7

For the value of n = 953

The left most digit is: 7

For the value of n = 963

The left most digit is: 7

For the value of n = 973

The left most digit is: 7

For the value of n = 1016

The left most digit is: 7

For the value of n = 1026

The left most digit is: 7

For the value of n = 1036

The left most digit is: 7

For the value of n = 1046

The left most digit is: 7

For the value of n = 1056

The left most digit is: 7

For the value of n = 1066

The left most digit is: 7

For the value of n = 1119

The left most digit is: 7

For the value of n = 1129

The left most digit is: 7

For the value of n = 1139

The left most digit is: 7

For the value of n = 1149

The left most digit is: 7

For the value of n = 1159

The left most digit is: 7

For the value of n = 1212

The left most digit is: 7

For the value of n = 1222

The left most digit is: 7

For the value of n = 1232

The left most digit is: 7

For the value of n = 1242

The left most digit is: 7

For the value of n = 1252

The left most digit is: 7

For the value of n = 1262

The left most digit is: 7

For the value of n = 1315

The left most digit is: 7

For the value of n = 1325

The left most digit is: 7

For the value of n = 1335

The left most digit is: 7

For the value of n = 1345

The left most digit is: 7

For the value of n = 1355

The left most digit is: 7

For the value of n = 1408

The left most digit is: 7

For the value of n = 1418

The left most digit is: 7

For the value of n = 1428

The left most digit is: 7

For the value of n = 1438

The left most digit is: 7

For the value of n = 1448

The left most digit is: 7

For the value of n = 1458

The left most digit is: 7

For the value of n = 1501

The left most digit is: 7

For the value of n = 1511

The left most digit is: 7

For the value of n = 1521

The left most digit is: 7

For the value of n = 1531

The left most digit is: 7

For the value of n = 1541

The left most digit is: 7

For the value of n = 1551

The left most digit is: 7

For the value of n = 1604

The left most digit is: 7

For the value of n = 1614

The left most digit is: 7

For the value of n = 1624

The left most digit is: 7

For the value of n = 1634

The left most digit is: 7

For the value of n = 1644

The left most digit is: 7

For the value of n = 1697

The left most digit is: 7

For the value of n = 1707

The left most digit is: 7

For the value of n = 1717

The left most digit is: 7

For the value of n = 1727

The left most digit is: 7

For the value of n = 1737

The left most digit is: 7

For the value of n = 1747

The left most digit is: 7

For the value of n = 1800

The left most digit is: 7

For the value of n = 1810

The left most digit is: 7

For the value of n = 1820

The left most digit is: 7

For the value of n = 1830

The left most digit is: 7

For the value of n = 1840

The left most digit is: 7

For the value of n = 1893

The left most digit is: 7

For the value of n = 1903

The left most digit is: 7

For the value of n = 1913

The left most digit is: 7

For the value of n = 1923

The left most digit is: 7

For the value of n = 1933

The left most digit is: 7

For the value of n = 1943

The left most digit is: 7

For the value of n = 1986

The left most digit is: 7

For the value of n = 1996

The left most digit is: 7

For the value of n = 2006

The left most digit is: 7

For the value of n = 2016

The left most digit is: 7

For the value of n = 2026

The left most digit is: 7

For the value of n = 2036

The left most digit is: 7

For the value of n = 2089

The left most digit is: 7

For the value of n = 2099

The left most digit is: 7

For the value of n = 2109

The left most digit is: 7

For the value of n = 2119

The left most digit is: 7

For the value of n = 2129

The left most digit is: 7

For the value of n = 2182

The left most digit is: 7

For the value of n = 2192

The left most digit is: 7

For the value of n = 2202

The left most digit is: 7

For the value of n = 2212

The left most digit is: 7

For the value of n = 2222

The left most digit is: 7

For the value of n = 2232

The left most digit is: 7

For the value of n = 2285

The left most digit is: 7

For the value of n = 2295

The left most digit is: 7

For the value of n = 2305

The left most digit is: 7

For the value of n = 2315

The left most digit is: 7

For the value of n = 2325

The left most digit is: 7

For the value of n = 2378

The left most digit is: 7

For the value of n = 2388

The left most digit is: 7

For the value of n = 2398

The left most digit is: 7

For the value of n = 2408

The left most digit is: 7

For the value of n = 2418

The left most digit is: 7

For the value of n = 2428

The left most digit is: 7

For the value of n = 2471

The left most digit is: 7

For the value of n = 2481

The left most digit is: 7

For the value of n = 2491

The left most digit is: 7

For the value of n = 2501

The left most digit is: 7

For the value of n = 2511

The left most digit is: 7

For the value of n = 2521

The left most digit is: 7

For the value of n = 2574

The left most digit is: 7

For the value of n = 2584

The left most digit is: 7

For the value of n = 2594

The left most digit is: 7

For the value of n = 2604

The left most digit is: 7

For the value of n = 2614

The left most digit is: 7

For the value of n = 2624

The left most digit is: 7

For the value of n = 2667

The left most digit is: 7

For the value of n = 2677

The left most digit is: 7

For the value of n = 2687

The left most digit is: 7

For the value of n = 2697

The left most digit is: 7

For the value of n = 2707

The left most digit is: 7

For the value of n = 2717

The left most digit is: 7

For the value of n = 2770

The left most digit is: 7

For the value of n = 2780

The left most digit is: 7

For the value of n = 2790

The left most digit is: 7

For the value of n = 2800

The left most digit is: 7

For the value of n = 2810

The left most digit is: 7

For the value of n = 2863

The left most digit is: 7

For the value of n = 2873

The left most digit is: 7

For the value of n = 2883

The left most digit is: 7

For the value of n = 2893

The left most digit is: 7

For the value of n = 2903

The left most digit is: 7

For the value of n = 2913

The left most digit is: 7

For the value of n = 2966

The left most digit is: 7

For the value of n = 2976

The left most digit is: 7

For the value of n = 2986

The left most digit is: 7

For the value of n = 2996

The left most digit is: 7

It means, the python has computed the values of 2^n for the value n from 1 to 3000 and given us the values of n for which 2^n starts with 7. For more values, we only need to change the range. To calculate the exponent of some other numbers, we will have to make the change in the second line where we defined the variable as number. If we take a close look at the values of n, we observe that the first six values of n are $46, 56, 66, 76, 86$ and 96. After that we get the values of n as $149, 159, 169, 179,$ and 189. Then the values of n are $242, 252, 262, 272, 282,$ and 292. After that the values of n are $345, 355, 365, 375,$ and 385. If we look at the sequences, we observe that there are 6 values of n, then there are 5 values of n, after that there are again 6 values of n and after that there are 5 values of n again. The difference between the successive values of n's are 10 and the gap between two sequences is 53. So, there is a hidden symmetry in arithmetic progression.

A.5 The values of n for which 2^n has first digit 5

If we run the code in A.3 for $n = 1$ to $n = 3000$, we will get the following results as exactly displayed in Python:

For the value of n = 9

The left most digit is: 5

For the value of n = 19

The left most digit is: 5

For the value of n = 29

The left most digit is: 5

For the value of n = 39

The left most digit is: 5

For the value of n = 49

The left most digit is: 5

For the value of n = 59

The left most digit is: 5

For the value of n = 69

The left most digit is: 5

For the value of n = 102

The left most digit is: 5

For the value of n = 112

The left most digit is: 5

For the value of n = 122

The left most digit is: 5

For the value of n = 132

The left most digit is: 5

For the value of n = 142

The left most digit is: 5

For the value of n = 152

The left most digit is: 5

For the value of n = 162

The left most digit is: 5

For the value of n = 172

The left most digit is: 5

For the value of n = 195

The left most digit is: 5

For the value of n = 205

The left most digit is: 5

For the value of n = 215

The left most digit is: 5

For the value of n = 225

The left most digit is: 5

For the value of n = 235

The left most digit is: 5

For the value of n = 245

The left most digit is: 5

For the value of n = 255

The left most digit is: 5

For the value of n = 265

The left most digit is: 5

For the value of n = 298

The left most digit is: 5

For the value of n = 308

The left most digit is: 5

For the value of n = 318

The left most digit is: 5

For the value of n = 328

The left most digit is: 5

For the value of n = 338

The left most digit is: 5

For the value of n = 348

The left most digit is: 5

For the value of n = 358

The left most digit is: 5

For the value of n = 391

The left most digit is: 5

For the value of n = 401

The left most digit is: 5

For the value of n = 411

The left most digit is: 5

For the value of n = 421

The left most digit is: 5

For the value of n = 431

The left most digit is: 5

For the value of n = 441

The left most digit is: 5

For the value of n = 451

The left most digit is: 5

For the value of n = 461

The left most digit is: 5

For the value of n = 494

The left most digit is: 5

For the value of n = 504

The left most digit is: 5

For the value of n = 514

The left most digit is: 5

For the value of n = 524

The left most digit is: 5

For the value of n = 534

The left most digit is: 5

For the value of n = 544

The left most digit is: 5

For the value of n = 554

The left most digit is: 5

For the value of n = 587

The left most digit is: 5

For the value of n = 597

The left most digit is: 5

For the value of n = 607

The left most digit is: 5

For the value of n = 617

The left most digit is: 5

For the value of n = 627

The left most digit is: 5

For the value of n = 637

The left most digit is: 5

For the value of n = 647

The left most digit is: 5

For the value of n = 657

The left most digit is: 5

For the value of n = 680

The left most digit is: 5

For the value of n = 690

The left most digit is: 5

For the value of n = 700

The left most digit is: 5

For the value of n = 710

The left most digit is: 5

For the value of n = 720

The left most digit is: 5

For the value of n = 730

The left most digit is: 5

For the value of n = 740

The left most digit is: 5

For the value of n = 750

The left most digit is: 5

For the value of n = 783

The left most digit is: 5

For the value of n = 793

The left most digit is: 5

For the value of n = 803

The left most digit is: 5

For the value of n = 813

The left most digit is: 5

For the value of n = 823

The left most digit is: 5

For the value of n = 833

The left most digit is: 5

For the value of n = 843

The left most digit is: 5

For the value of n = 876

The left most digit is: 5

For the value of n = 886

The left most digit is: 5

For the value of n = 896

The left most digit is: 5

For the value of n = 906

The left most digit is: 5

For the value of n = 916

The left most digit is: 5

For the value of n = 926

The left most digit is: 5

For the value of n = 936

The left most digit is: 5

For the value of n = 946

The left most digit is: 5

For the value of n = 979

The left most digit is: 5

For the value of n = 989

The left most digit is: 5

For the value of n = 999

The left most digit is: 5

For the value of n = 1009

The left most digit is: 5

For the value of n = 1019

The left most digit is: 5

For the value of n = 1029

The left most digit is: 5

For the value of n = 1039

The left most digit is: 5

For the value of n = 1072

The left most digit is: 5

For the value of n = 1082

The left most digit is: 5

For the value of n = 1092

The left most digit is: 5

For the value of n = 1102

The left most digit is: 5

For the value of n = 1112

The left most digit is: 5

For the value of n = 1122

The left most digit is: 5

For the value of n = 1132

The left most digit is: 5

For the value of n = 1142

The left most digit is: 5

For the value of n = 1165

The left most digit is: 5

For the value of n = 1175

The left most digit is: 5

For the value of n = 1185

The left most digit is: 5

For the value of n = 1195

The left most digit is: 5

For the value of n = 1205

The left most digit is: 5

For the value of n = 1215

The left most digit is: 5

For the value of n = 1225

The left most digit is: 5

For the value of n = 1235

The left most digit is: 5

For the value of n = 1268

The left most digit is: 5

For the value of n = 1278

The left most digit is: 5

For the value of n = 1288

The left most digit is: 5

For the value of n = 1298

The left most digit is: 5

For the value of n = 1308

The left most digit is: 5

For the value of n = 1318

The left most digit is: 5

For the value of n = 1328

The left most digit is: 5

For the value of n = 1338

The left most digit is: 5

For the value of n = 1361

The left most digit is: 5

For the value of n = 1371

The left most digit is: 5

For the value of n = 1381

The left most digit is: 5

For the value of n = 1391

The left most digit is: 5

For the value of n = 1401

The left most digit is: 5

For the value of n = 1411

The left most digit is: 5

For the value of n = 1421

The left most digit is: 5

For the value of n = 1431

The left most digit is: 5

For the value of n = 1464

The left most digit is: 5

For the value of n = 1474

The left most digit is: 5

For the value of n = 1484

The left most digit is: 5

For the value of n = 1494

The left most digit is: 5

For the value of n = 1504

The left most digit is: 5

For the value of n = 1514

The left most digit is: 5

For the value of n = 1524

The left most digit is: 5

For the value of n = 1557

The left most digit is: 5

For the value of n = 1567

The left most digit is: 5

For the value of n = 1577

The left most digit is: 5

For the value of n = 1587

The left most digit is: 5

For the value of n = 1597

The left most digit is: 5

For the value of n = 1607

The left most digit is: 5

For the value of n = 1617

The left most digit is: 5

For the value of n = 1627

The left most digit is: 5

For the value of n = 1650

The left most digit is: 5

For the value of n = 1660

The left most digit is: 5

For the value of n = 1670

The left most digit is: 5

For the value of n = 1680

The left most digit is: 5

For the value of n = 1690

The left most digit is: 5

For the value of n = 1700

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For the value of n = 1710

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For the value of n = 1720

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For the value of n = 1753

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For the value of n = 1763

The left most digit is: 5

For the value of n = 1773

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For the value of n = 1783

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For the value of n = 1793

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For the value of n = 1803

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For the value of n = 1813

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For the value of n = 1823

The left most digit is: 5

For the value of n = 1846

The left most digit is: 5

For the value of n = 1856

The left most digit is: 5

For the value of n = 1866

The left most digit is: 5

For the value of n = 1876

The left most digit is: 5

For the value of n = 1886

The left most digit is: 5

For the value of n = 1896

The left most digit is: 5

For the value of n = 1906

The left most digit is: 5

For the value of n = 1916

The left most digit is: 5

For the value of n = 1949

The left most digit is: 5

For the value of n = 1959

The left most digit is: 5

For the value of n = 1969

The left most digit is: 5

For the value of n = 1979

The left most digit is: 5

For the value of n = 1989

The left most digit is: 5

For the value of n = 1999

The left most digit is: 5

For the value of n = 2009

The left most digit is: 5

For the value of n = 2042

The left most digit is: 5

For the value of n = 2052

The left most digit is: 5

For the value of n = 2062

The left most digit is: 5

For the value of n = 2072

The left most digit is: 5

For the value of n = 2082

The left most digit is: 5

For the value of n = 2092

The left most digit is: 5

For the value of n = 2102

The left most digit is: 5

For the value of n = 2112

The left most digit is: 5

For the value of n = 2135

The left most digit is: 5

For the value of n = 2145

The left most digit is: 5

For the value of n = 2155

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For the value of n = 2165

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For the value of n = 2175

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For the value of n = 2185

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For the value of n = 2195

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For the value of n = 2205

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For the value of n = 2238

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For the value of n = 2248

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For the value of n = 2258

The left most digit is: 5

For the value of n = 2268

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For the value of n = 2278

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For the value of n = 2288

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For the value of n = 2298

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For the value of n = 2308

The left most digit is: 5

For the value of n = 2331

The left most digit is: 5

For the value of n = 2341

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For the value of n = 2351

The left most digit is: 5

For the value of n = 2361

The left most digit is: 5

For the value of n = 2371

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For the value of n = 2381

The left most digit is: 5

For the value of n = 2391

The left most digit is: 5

For the value of n = 2401

The left most digit is: 5

For the value of n = 2434

The left most digit is: 5

For the value of n = 2444

The left most digit is: 5

For the value of n = 2454

The left most digit is: 5

For the value of n = 2464

The left most digit is: 5

For the value of n = 2474

The left most digit is: 5

For the value of n = 2484

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For the value of n = 2494

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For the value of n = 2527

The left most digit is: 5

For the value of n = 2537

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For the value of n = 2547

The left most digit is: 5

For the value of n = 2557

The left most digit is: 5

For the value of n = 2567

The left most digit is: 5

For the value of n = 2577

The left most digit is: 5

For the value of n = 2587

The left most digit is: 5

For the value of n = 2597

The left most digit is: 5

For the value of n = 2630

The left most digit is: 5

For the value of n = 2640

The left most digit is: 5

For the value of n = 2650

The left most digit is: 5

For the value of n = 2660

The left most digit is: 5

For the value of n = 2670

The left most digit is: 5

For the value of n = 2680

The left most digit is: 5

For the value of n = 2690

The left most digit is: 5

For the value of n = 2723

The left most digit is: 5

For the value of n = 2733

The left most digit is: 5

For the value of n = 2743

The left most digit is: 5

For the value of n = 2753

The left most digit is: 5

For the value of n = 2763

The left most digit is: 5

For the value of n = 2773

The left most digit is: 5

For the value of n = 2783

The left most digit is: 5

For the value of n = 2793

The left most digit is: 5

For the value of n = 2816

The left most digit is: 5

For the value of n = 2826

The left most digit is: 5

For the value of n = 2836

The left most digit is: 5

For the value of n = 2846

The left most digit is: 5

For the value of n = 2856

The left most digit is: 5

For the value of n = 2866

The left most digit is: 5

For the value of n = 2876

The left most digit is: 5

For the value of n = 2886

The left most digit is: 5

For the value of n = 2919

The left most digit is: 5

For the value of n = 2929

The left most digit is: 5

For the value of n = 2939

The left most digit is: 5

For the value of n = 2949

The left most digit is: 5

For the value of n = 2959

The left most digit is: 5

For the value of n = 2969

The left most digit is: 5

For the value of n = 2979

The left most digit is: 5

A.6 The values of n for which 2^n has first digit 9

If we run the code in A.3 for $n = 1$ to $n = 3000$, we will get the following results as exactly displayed in Python:

For the value of n = 53

The left most digit is: 9

For the value of n = 63

The left most digit is: 9

For the value of n = 73

The left most digit is: 9

For the value of n = 83

The left most digit is: 9

For the value of n = 93

The left most digit is: 9

For the value of n = 156

The left most digit is: 9

For the value of n = 166

The left most digit is: 9

For the value of n = 176

The left most digit is: 9

For the value of n = 186

The left most digit is: 9

For the value of n = 249

The left most digit is: 9

For the value of n = 259

The left most digit is: 9

For the value of n = 269

The left most digit is: 9

For the value of n = 279

The left most digit is: 9

For the value of n = 289

The left most digit is: 9

For the value of n = 352

The left most digit is: 9

For the value of n = 362

The left most digit is: 9

For the value of n = 372

The left most digit is: 9

For the value of n = 382

The left most digit is: 9

For the value of n = 445

The left most digit is: 9

For the value of n = 455

The left most digit is: 9

For the value of n = 465

The left most digit is: 9

For the value of n = 475

The left most digit is: 9

For the value of n = 485

The left most digit is: 9

For the value of n = 548

The left most digit is: 9

For the value of n = 558

The left most digit is: 9

For the value of n = 568

The left most digit is: 9

For the value of n = 578

The left most digit is: 9

For the value of n = 641

The left most digit is: 9

For the value of n = 651

The left most digit is: 9

For the value of n = 661

The left most digit is: 9

For the value of n = 671

The left most digit is: 9

For the value of n = 734

The left most digit is: 9

For the value of n = 744

The left most digit is: 9

For the value of n = 754

The left most digit is: 9

For the value of n = 764

The left most digit is: 9

For the value of n = 774

The left most digit is: 9

For the value of n = 837

The left most digit is: 9

For the value of n = 847

The left most digit is: 9

For the value of n = 857

The left most digit is: 9

For the value of n = 867

The left most digit is: 9

For the value of n = 930

The left most digit is: 9

For the value of n = 940

The left most digit is: 9

For the value of n = 950

The left most digit is: 9

For the value of n = 960

The left most digit is: 9

For the value of n = 970

The left most digit is: 9

For the value of n = 1033

The left most digit is: 9

For the value of n = 1043

The left most digit is: 9

For the value of n = 1053

The left most digit is: 9

For the value of n = 1063

The left most digit is: 9

For the value of n = 1126

The left most digit is: 9

For the value of n = 1136

The left most digit is: 9

For the value of n = 1146

The left most digit is: 9

For the value of n = 1156

The left most digit is: 9

For the value of n = 1219

The left most digit is: 9

For the value of n = 1229

The left most digit is: 9

For the value of n = 1239

The left most digit is: 9

For the value of n = 1249

The left most digit is: 9

For the value of n = 1259

The left most digit is: 9

For the value of n = 1322

The left most digit is: 9

For the value of n = 1332

The left most digit is: 9

For the value of n = 1342

The left most digit is: 9

For the value of n = 1352

The left most digit is: 9

For the value of n = 1415

The left most digit is: 9

For the value of n = 1425

The left most digit is: 9

For the value of n = 1435

The left most digit is: 9

For the value of n = 1445

The left most digit is: 9

For the value of n = 1455

The left most digit is: 9

For the value of n = 1518

The left most digit is: 9

For the value of n = 1528

The left most digit is: 9

For the value of n = 1538

The left most digit is: 9

For the value of n = 1548

The left most digit is: 9

For the value of n = 1611

The left most digit is: 9

For the value of n = 1621

The left most digit is: 9

For the value of n = 1631

The left most digit is: 9

For the value of n = 1641

The left most digit is: 9

For the value of n = 1704

The left most digit is: 9

For the value of n = 1714

The left most digit is: 9

For the value of n = 1724

The left most digit is: 9

For the value of n = 1734

The left most digit is: 9

For the value of n = 1744

The left most digit is: 9

For the value of n = 1807

The left most digit is: 9

For the value of n = 1817

The left most digit is: 9

For the value of n = 1827

The left most digit is: 9

For the value of n = 1837

The left most digit is: 9

For the value of n = 1900

The left most digit is: 9

For the value of n = 1910

The left most digit is: 9

For the value of n = 1920

The left most digit is: 9

For the value of n = 1930

The left most digit is: 9

For the value of n = 1940

The left most digit is: 9

For the value of n = 2003

The left most digit is: 9

For the value of n = 2013

The left most digit is: 9

For the value of n = 2023

The left most digit is: 9

For the value of n = 2033

The left most digit is: 9

For the value of n = 2096

The left most digit is: 9

For the value of n = 2106

The left most digit is: 9

For the value of n = 2116

The left most digit is: 9

For the value of n = 2126

The left most digit is: 9

For the value of n = 2189

The left most digit is: 9

For the value of n = 2199

The left most digit is: 9

For the value of n = 2209

The left most digit is: 9

For the value of n = 2219

The left most digit is: 9

For the value of n = 2229

The left most digit is: 9

For the value of n = 2292

The left most digit is: 9

For the value of n = 2302

The left most digit is: 9

For the value of n = 2312

The left most digit is: 9

For the value of n = 2322

The left most digit is: 9

For the value of n = 2385

The left most digit is: 9

For the value of n = 2395

The left most digit is: 9

For the value of n = 2405

The left most digit is: 9

For the value of n = 2415

The left most digit is: 9

For the value of n = 2425

The left most digit is: 9

For the value of n = 2488

The left most digit is: 9

For the value of n = 2498

The left most digit is: 9

For the value of n = 2508

The left most digit is: 9

For the value of n = 2518

The left most digit is: 9

For the value of n = 2581

The left most digit is: 9

For the value of n = 2591

The left most digit is: 9

For the value of n = 2601

The left most digit is: 9

For the value of n = 2611

The left most digit is: 9

For the value of n = 2621

The left most digit is: 9

For the value of n = 2684

The left most digit is: 9

For the value of n = 2694

The left most digit is: 9

For the value of n = 2704

The left most digit is: 9

For the value of n = 2714

The left most digit is: 9

For the value of n = 2777

The left most digit is: 9

For the value of n = 2787

The left most digit is: 9

For the value of n = 2797

The left most digit is: 9

For the value of n = 2807

The left most digit is: 9

For the value of n = 2870

The left most digit is: 9

For the value of n = 2880

The left most digit is: 9

For the value of n = 2890

The left most digit is: 9

For the value of n = 2900

The left most digit is: 9

For the value of n = 2910

The left most digit is: 9

For the value of n = 2973

The left most digit is: 9

For the value of n = 2983

The left most digit is: 9

For the value of n = 2993

The left most digit is: 9

Bibliography

[1] V. I. Arnold *Mathematical Methods of Classical Mechanics*. .

[2] Mark Levi. *Classical mechanics with Calculus of variations and Optimal control – an intuitive introduction*.

[3] Mark Levi. *The Mathematical Mechanic*.

[4] Benford's Law.
https://en.wikipedia.org/wiki/Benford%27s_law

[5] Lagrangian Mechanics.
https://en.wikipedia.org/wiki/Lagrangian_mechanics