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Intuition on Trial

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INTUITION ON TRIAL

by

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Signatures of the Members of the Examining Committee:
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The predominant characteristic of the mathematician, as the writer sees him, is that of unrest, for the mathematician is never fully satisfied. To illustrate, he accepts the challenge offered by a problem new to him. Until he sees the solution to the problem in entirety, he is tense, anxious. After the problem has been solved, calm is restored and the thrill of accomplishment supplants the former tenseness. Offhand, then, it would appear that the mathematician at this stage is satisfied and can lean back in his chair, contented. Before long, however, he meets another problem, accepts the challenge, and again feels tensions which will be resolved only when the problem is resolved in his own mind. He begins to suspect that after the present challenge there will be another, and then another.

What, then, leads the mathematician on? What does he hope to accomplish? Why does awareness of a new problem-situation invite consideration on his part, rather than rejection? What is the mathematician trying to do?

In the first place, a problem by its very nature suggests an incompleteness, a lack, something missing. The mathematician, discontented with puzzles not completely fitted together, finds himself searching for a pattern in order that he may discover the missing pieces. Not happy with isolated statements, he strives to establish relationships among the statements. Finally, by inferences drawn from the relationships discovered, he sees
the solution and consequently the entire pattern. An admiration for "wholeness" then, partly explains the drive activating the mathematician.

In addition, the mere realization of the presence of a challenge invites action from anyone so geared. That is, such a person feels that he must accept the challenge presented to him. To ignore the challenge is unthinkable. Similarly, the story is told that a mountain-climber, asked to analyze his reasons for scaling a particular mountain, explained that he had to climb the mountain simply because the mountain was there. The presence of the mountain constituted the challenge, finding the best manner in which to scale the mountain was the problem, and making the actual climb would prove his acceptance of the challenge.

Besides revealing a yearning for "wholeness" and a desire to meet headlong any challenging situation, the mathematician shows a keen responsiveness to beauty. For one thing, he finds beauty in a perfectly constructed proof. True, the ordered symmetry of the proof may not be at all apparent to one whose eye and mind are not so attuned. However, lack of awareness of beauty on the part of some people does not mean the proof holds no attractiveness for anyone. Thus, one piece of land may belong to a Mr. Jackson, the adjoining piece may belong to a Mr. Reeves, but the landscape belongs to whoever has the eye to behold it.

The mathematician sees beauty not only in the proof before him, but also in the reasoning which has produced the proof. Clear-cut ideas, precisely and forcefully
expressed, are interesting in themselves, thereby attracting the mathematician's attention and evoking his admiration.

However, the aspect of the mathematician's life that is probably most often slighted in appraisals is his creativeness. For instance, in solving a problem or in proving a theorem he has constructed something, a whole out of what were disorganized segments. By examining the segments, by visualizing a pattern connecting them, and by supplying missing parts, he has built a structure, an idea resting on a firm foundation of logical reasoning. Moreover, the idea so structured is acceptable, indeed irresistible, to anyone subscribing to the assumptions on which the reasoning is based. He may express his creativeness in attacking a familiar relationship from a new point of view; he may pose a challenging problem to his students, his colleagues, or his family; or, if he is one of a privileged few, he may have the happy experience of building up by induction a theorem heretofore not guessed by anyone.

Whether the mathematician's efforts are examined for aspects of "wholeness", beauty, or creativeness, one fact must surely be noted. In common with the work of scholars in other fields, his work evidences a search for truth, or rather, a search for what he conceives to be truth.

In his attempts to discover truth, though, he realizes that he will never fully succeed. After all, as that discerning mathematician, Edson H. Taylor, has remarked,
"The search for truth is an infinite process." The mathematician, teased by a question that will not leave him alone, applies resources of ingenuity, concentration, and far-sightedness to getting the answer. By the time the one question has been dealt with to his satisfaction, other questions have arisen, each just as insistent as the first.

Sometimes, in his search for answers, he has had the thrill of getting sudden, totally unexpected insight into the problem at hand. That is, he may have grasped immediately the very relationship pointing to the solution, when he had not previously worked out the pertinent steps. Then too, it is just possible that, when involved in proving a theorem, he has seen in a "flash", so to speak, what step he should take next. Since the mathematician attributes both of these exciting flashes of understanding to intuition, he may, in the course of his adventures, find himself wondering about intuition and subsequently hunting for explanations.
Chapter I
Introduction to Intuition

Down through the ages, in all places and at all times, man has been seeking truth. At times truth has stood maddeningly close, yet has eluded capture. At other times, despite endless efforts with test-tube and beaker, with Geiger counter and atom-smasher, it has stood aloof, unreachable. On still other occasions, for brief moments, it has appeared tractable, even attainable.

Thus history has witnessed Abraham following the direction of an invisible god into a promised land, Euclid applying with vigor his powers of organization, and Omar Khayyam, despite emphasis on the here-and-now, looking for order in algebra. Still later arises Spinoza, who, in his seclusion, searches for answers to the insistent question of what constitutes knowledge.

In addition to considering what is true, man has been teased by the puzzle of explaining how he knows a particular belief to be true. Accordingly, he may ascribe an idea to intuition. That is, he may have had the experience of "knowing", when he had not previously come across evidence that would "prove" his idea to be true. Perhaps now is the time to bring out a point often overlooked, that the lack of proof does not necessarily indicate that a statement is false, or even doubtful. As an extreme example, it is conceivable that a person might be unable to supply any testimony or record of his birth. Thus he could not prove, in a legal sense, that he had
ever been born. A few years ago a Syrian lecturer lampooned the American emphasis on birth certificates. Apparently in Syria the mere presence of the person was considered evidence enough of the fact of birth.

In addition to intuition there are three other ways of knowing, which will be examined in a moment. Let us think of the ways of knowing as the rungs of a ladder with the most honored, reliable method at the top. Obviously proof occupies the topmost rung. At the other extreme, and there may well be argument, we are going to place intuition. No attempt will be made to assign authority and experiment to definite rungs, as the dependability of either of these hinges upon the care with which it is used.

For instance, upon examining authority as a method of knowing facts, probably the most serious consideration lies in what one is willing to call an authority. What are his credentials? Is he recognized as an authority by people whose opinion can be respected? Is he a "self-styled" authority? The fact that his statements have been published does not make him an authority. Yet frequently gullible people are willing to accept the most outlandish claims, simply because such claims have been published. Such people will believe everything from the label on a patent-medicine bottle to an august pronouncement that the world is about to end.

Furthermore, the fact that a person speaks or writes in a self-assured manner does not make him an authority.
Indeed, much self-assurance results from too little information, or from knowledge about only one aspect of a subject. Even the fact that a man is regarded highly by the scholars of the day as a recognized authority does not mean that he is in possession of absolute truth. For example, Aristotle was for centuries deemed the source of information on anatomy and almost any other subject as well. According to legend, if a mediaeval teacher were dissecting a dog for the benefit of his anatomy class, and if one of the dog's organs did not comply with what was written in Aristotle's book, the dog was wrong!

Another limitation to using authority as a way to support arguments is the fact that although a man may have convinced himself that the authority he is naming really is such, the person with whom he is debating may not accept that same source as reliable. Consider for a moment the task of trying to convince a Buddhist of Christian beliefs by quoting the Bible. Apparently, for any individual, an authority is any source that, for various reasons, he accepts as sound; consequently, he uses the source both to gain new information and to support opinions that to him seem valid.

As to the matter of experiment, there seem to be at least four weaknesses or limitations. First, and this requirement surely could not be over-stressed, is the condition that the experimenter must assure himself that he is controlling all but one factor. Otherwise he cannot know
that the result obtained has stemmed from the one condition for which he is testing. Another difficulty lies in attempting to duplicate conditions in a series of experiments. If the experimenter is not extremely cautious in this respect, he will be unable safely to draw a generalization from his result. In the third place, there is the liability of his concluding more than what has actually been established. As an illustration, suppose a hybrid black guinea pig were mated to another hybrid black one. On authority of those who have conducted the experiment, there is a strong probability that one-fourth of the offspring will be white, the others, black. Suppose a man not conversant with the laws of heredity were to look at the two black parents, look at the offspring, regard especially the black ones, and conclude that all the black ones were hybrid with respect to color of coat. Such a conclusion would appear "reasonable", as the presence of the white offspring would lead him to think that the black ones could not be pure-line black. However, this is a false conclusion, for actually any one of the black offspring might be pure-line black, a point that would be settled if the study were carried through to the extent of crossing a black one with one that was known to be hybrid with respect to color. The occurrence of no white ones in the third generation thus produced would indicate definitely that the questionable parent was pure-line. Thus, to get back to the original pair considered, the
conclusion drawn by the man was unsound. In summation, all that he had actually established was that crossing a particular black male guinea pig with a particular black female guinea pig yielded offspring, some of which were black, others, white. That one fact was all that he had ascertained as a result of that particular experiment.

A fourth limitation in the use of experiment is the obvious fact that one cannot test all cases. At first glance, some may be surprised at the mention of anything so readily apparent. However, as one cannot test all cases having like conditions, one cannot determine the results for all cases. Hence one cannot know through experimentation that a certain set of conditions yields a certain result for all cases.

Although it is impossible to test all cases, it is possible, nevertheless, to establish the validity of a formula by the process of mathematical induction. This process, which consists of only two steps, will be reviewed by making reference to the formula involving compound interest. The formula, \( A = P(1 + r)^n \), gives the amount \( A \) to which an original principal \( P \) would accumulate in \( n \) years if invested at an interest rate \( r \), compounded annually. Since the interest is compounded, there is a new principal at the beginning of each successive year. It is easy to see that, at the beginning of the second year, the new principal will be \( (P + Pr) \), which equals
P(l + r). Thus, at the beginning of the third year, the new principal will be \( P(l + r) + P(l + r)r \), which, when simplified, becomes \( P(l + r)^2 \). By a similar argument, the new principal at the beginning of the fourth year can be shown to be equal to \( P(l + r)^3 \). However, the formula has been tested only in those cases in which \( n \) is equal to 1, 2, or 3. It is unsafe to assume at this juncture that the formula will hold for all integral \( n \).

The solution to this dilemma lies in applying the process that is termed mathematical induction. First of all, the formula is assumed to be valid for some value of \( n \), for instance, for \( n \) equal to \( s \). The immediate goal is to show the existence of an inheritance property, namely, that the validity of the formula for \( n \) equal to \( s \) implies the validity of the corresponding formula for the succeeding integral value, \( s + 1 \). Thus, we assume that at the end of \( s \) years, the accumulation is \( P(l + r)^s \). Since, during the next year, the interest earned will be \( P(l + r)^s r \), at the end of \((s + 1)\) years, the total amount can be shown to be \( P(l + r)^{s+1} \), which establishes the inheritance property. Once the all-important inheritance property has been established, one can assume that, if the formula is verified for some specific value of \( n \), usually for \( n \) equal to 1, then the formula is likewise true for the case in which \( n \) is equal to the next successive integer. Of course, this last statement is tantamount to saying that the formula holds for all \( n \). Usually, however, the verification for
a specific value is established first; then, the inheritance property is shown to exist. Hence, although not all cases can be tested, certain formulas can be established by mathematical induction.

In general, the validity of any proposition is recognized most readily, if the proposition has been proved. Built upon a foundation of certain statements deemed acceptable, a proof proceeds along the lines decreed by logic, step by step, to an unequivocal conclusion. Thus, he who subscribes to the statements on which the proof is constructed is necessarily convinced of the "truth" of the conclusion. Characterized by a high degree of reliability and certainty, therefore, the method of proof deservedly occupies the position of greatest honor, the topmost rung. When using this method, one must be sure, if the proof is to be rigorous, that every step can be supported by something previously accepted. That is, each step must be substantiated by a proof already established, by a definition, or by an assumption, together with correct application of the rules pertaining to logical inference. Moreover, even before attempting to outline the proof, the student meets a sizable hurdle. That is, he has the responsibility of seeing to it that he does not introduce special conditions. To put it differently, he must determine that he does not assume any conditions other than those specifically stated by the hypotheses. This careless intrusion of special conditions is particularly dangerous at the stage when he starts
to make a drawing. For instance, if the hypotheses of a theorem in tenth-grade geometry mention a parallelogram, the student, in making an illustration, must not draw a rectangle, as the rectangle is a special type of parallelogram. If the student were to draw a rectangle, he would be assuming that he was to deal with perpendicular lines and ninety-degree angles. That is, he would be guilty of having intruded special conditions, conditions neither explicitly stated nor even implied in the hypotheses.

In these remarks, it is hoped that a groundwork for the study of intuition has been laid. In laying such groundwork, we have presented intuition as one of four ways of knowing. Having dealt with the limitations of three of the ways, we shall venture next into a discussion of the mysteries of intuition itself.
Chapter II
Characteristics of Intuition

Probably the most formidable deterrent to the study of intuition is the difficulty of definition. At the outset the impossibility of defining the term with exactness is asserted; arguments to support this assertion will now be outlined. Later, the characteristics of intuition will be noted and examined in some detail.

For one thing, the abstractness of the term makes precise definition difficult. Invisible, inaudible, and most certainly intangible, intuition does not lend itself to the type of detailed observation possible with concrete objects. That is, the student of intuition cannot examine his subject as he might inspect, for instance, a picture, a musical composition, or a rose. Since the properties are of an elusive quality, they prevent proper classification, a prerequisite to definition. Of course, abstractness alone does not imply the impossibility of definition, but surely at best such a condition makes the task of defining an imposing one.

A far stronger argument against definition lies in the fact that what is intuitively obvious to one may not be, to another. That is, if the application of a concept is familiar to a man, he is apt to give that concept

intuitive status. A fact intuitively clear to a mechanic might be known to another only as a conclusion derived from the most involved reasoning. A relationship grasped immediately by a mathematician might never be suspected by another. An observation ascribed to intuition by a botanist might well be questioned by the uninitiated.

Since there is room for vast disagreement as to just what ideas may be known by intuition, it follows that intuition itself cannot be categorized or pigeon-holed in an unimpeachable manner. Hence the term intuition defies definition.

Although the term remains undefined, or perhaps because of the very lack of precise definition, it is imperative that intuition at least be described. Accordingly, as intimated in the first paragraph, several characteristics will be viewed.

In the first place, intuition is not diametrically opposed to reason. For example, the personnel manager of an industrial firm finds himself confronted with eleven applications for a particular job. He will conduct a short interview with each applicant with the object of eliminating eight of the hopefuls. At almost the first glance, and after the slightest exchange of words, he will have judged which eight will be unsuited to the work. Intuition has at least helped in forming the first judgments. That is, the man "feels" that a particular three will be
well-suited to answering the demands of the job. At the next stage, the most careful reasoning must be employed. The personnel manager must sort through several different kinds of evidence, such as the ratings of former employers, judgments pertaining to the applicant's character, and considered opinions as to personality traits, such as promptness, dependability, initiative, and cooperativeness. All evidence he must weigh carefully in the hopes of choosing the best of the remaining three. Thus intuition plus careful reasoning guide his selection.

As another instance, a student has been assigned a theorem to prove. He observes not only all facts given in the hypothesis, but also the relationship he is to prove. By reasoning, he determines the validity of each step, but which step to take next is often indicated by intuition. Again, logical reasoning has worked alongside intuition to produce results.

Although intuition and logical reasoning may operate simultaneously in the solution of a problem, intuition can be described as a way of getting ideas without words, a short cut to knowledge. Understandably then, a person challenged to support a view known intuitively will be at a loss for words to explain how he knows.

Furthermore, intuition is possibly almost the only way of reasoning employed by those people who have given no
study to the subject of reaching conclusions. Those who have never given thought to deduction as a way of drawing inferences continually make "snap judgments", not necessarily false; seemingly, such persons do not fall back on logic to defend their views.

The reference just made to the fact that "snap judgments" are not always incorrect suggests a point that can hardly be stressed too much, namely, that intuitive reasoning is not necessarily weak or faulty. For instance, there is the trite case of the wife who warns her husband against having business dealings with a man she has just met. When pressed for reasons, she asserts only that she knows the would-be business associate cannot be trusted. Of course, the concluding portion of such a tale always depicts the wife's predictions fulfilled and the husband's undying gratitude for having been warned in time.

In addition, intuition is usually a satisfactory guide in reaching conclusions to the extent that things seem to be as they are. Thus the student, upon inspecting a pair of intersecting straight lines for the first time, intuitively concludes that the vertical angles are equal. The vertical angles appear to be equal; they are equal.

1. The phrase "intuitive reasoning" may be paradoxical to some; that is, some observers may with vehemence claim that if a fact be intuitively known, then that fact cannot have stemmed from reasoning. If the term reasoning gives difficulty, perhaps intuition can be thought of as a faculty of the human mind which acts outside the realm of consciousness.
In situations in which appearances are deceiving, however, intuition can mislead, as in the case of optical illusions. A notable instance of such misleading is found in the belief formerly held that the earth was flat. Such a view of the earth is reasonably consistent with what one observes and is therefore understandable. Hence intuition is reliable only in those instances in which a situation appears to be the way it actually is.

All knowledge gained intuitively has been obtained in one of two ways, either by sense intuition or by intellectual intuition. In this discussion, sense intuition shall mean a situation such as the following. With reference again to the two intersecting straight lines, a person regarding the lines realizes the vertical angles formed are equal. The fact that such a relationship can be proved does not add to sureness. His realization that the vertical angles are equal is a fact known by sense intuition, the sense concerned being that of sight. It has been found that many people limit the use of the word intuition to designating the sense type only.

An example of intellectual intuition, as the term is understood by the writer, can be found in problem-solving. At a certain step in the work, the problem-solver, upon examining the data already assembled, sees two possible avenues to follow. Without resorting to trial-and-error, he decides his plan of action. Such a plan, it seems, can
be said to be known by intellectual intuition, for none of his senses have played a major part in his reaching a decision.

It is only natural that the lack of precise definition has resulted in a confusion regarding the use of the term, intuition. Since the word is apparently frequently misused, a discussion of the kinds of misuse may, by contrast, clarify correct usage. We are alluding to the practice of using the word intuition as a cover-up for a belief or a prejudice that cannot actually be proved. A case in point concerns an incident, factual, by the way, in which a lady sought to reinforce her belief in the superiority of the white race over other races by stating that such a view was intuitively held and obviously valid. When asked to supply documentary evidence, all she could do was to fall back on the usual meaningless cliches plus her personal interpretation of intuition.

A similar instance of misuse deals with the employer who steadfastly refuses to hire what he terms a foreigner. It may be that to him, a foreigner is anyone whose name is not Anglo-Saxon. However that may be, when questioned on his policies, he asserts vehemently that the wisdom of his action should be clear by intuition. Clearly he is implying that, if anyone questions his viewpoint, the fault cannot lie in the employer's attitude, but rather in the challenger's intuitive powers.

Although there assuredly are instances when intuition
may be at fault, there is encouragement in the thought that intuitive powers can possibly be improved. Some do not accept the possibility of such improvement; such skeptics may consider their position somewhat weakened by the account of a class conducted by the General Electric Company several years ago. Enrolled in a course called "practical engineering", the twenty college graduates spent class sessions telling one another their hunches, just as they thought of them. The act of explaining their ideas to others helped clarify their own thinking, initiated by the hunches, which, in turn, were suggested by intuition. Although class sessions were characterized by discussion, the time outside class was characterized by action, and plenty of it. That is, outside the classroom, the students performed whatever operations were indicated in order to carry out their hunches. The tangible result of the course was a formidable number of inventions useful to the company. Moreover, the accomplishments of the class are all the more interesting in view of the fact that some of those enrolled had never invented anything before.

Reactions to this account are bound to vary greatly. The sponsors of the class enthusiastically point out the imposing array of "first inventions". According to the sponsors, the students' inventions were based on intuitively-obtained hunches. The production of such inventions proves, they reason, that the methods of conducting the

1. C. G. Suits, "Heed That Hunch", American Magazine (December, 1945), CXL, 142.
class had developed or broadened the students' intuitive powers. Their opponents might claim, on the other hand, that the sponsors had succumbed to the "post hoc" fallacy. It is just possible that those enrolled in the class would have carried out their inventions, even if they had not ever been members of the "practical engineering" class. In other words, perhaps the sponsors had cited a cause-effect relationship that was not valid.

Those who accept the premise that a broadening of intuitive powers is possible will naturally wonder just what factors are conducive to such broadening. More explicitly, what traits or habits can be developed in order to achieve such a goal? One writer suggests, among other things, training oneself in alertness, sensitivity, and discipline of mind. The mention of these three undoubtedly worthy aids makes the improvement of intuitive powers sound far from an easy task. Another observer is more specific, for he says outright that one's interests should be broadened, in order that intuition can be supplied with plenty of material. Seemingly, the more varied the experiences, the greater the possibility of insight into a given situation.

Furthermore, one should learn not to distrust a new idea simply because it is foreign; rather, one should welcome the new idea. As an illustration, a student, after several


2. C. G. Suits, "Heed That Hunch", American Magazine (December, 1945), CXL, 143.
unsuccessful bouts with a formidable problem, may be on the verge of despair. Perhaps at that very moment a radically different approach occurs to him. If he is wise, he will not summarily dismiss the new thought as ridiculous or illogical. After all, in order to solve his problem, he may need an entirely new view; perhaps he needs an "overview". On the chance that his new slant will lead to the insight requisite for success, he should study the ramifications of the new idea. At the very least, he needs to consider as many aspects of the idea as are necessary to indicate definitely whether or not the idea is practicable.

The last suggestion for the improvement of intuitive powers to be made is of unquestionable validity, as it has been practiced so often by so many different people. When confronted with a problem which defies solution, the problem-solver should write down all the details as far as he can go. Then he should lay the problem aside for a while. During the "cooling" period, while the problem-solver is engaged in some radically different activity, he may get complete, full insight into the problem.

In this chapter, we have not only given arguments against defining the term intuition, but we have also described several characteristics of intuition. Although the term is not precisely defined, this paper will be consistent in the way in which the word is used. Throughout, intuition shall be used in relation to situations in which facts are directly known, without recourse to logical reasoning.
Chapter III
Intuition and Problem-solving

To the mathematician, the facet of intuition that is probably most conspicuous is the role played in problem-solving. An integral part of the mathematician's activities, the quest for solutions to problems appears to be a dominant feature of other disciplines as well. Thus the physicist, the chemist, the zoologist, and the geographer owe at least some of the developments in their fields to the existence of problems, or rather, to the fact that certain problems have been tackled and resolved. Perhaps it is not too sweeping a statement to assert that every type of progress experienced by the human race has resulted from someone's awareness of a problem.

Since problems arise continually, not only in fields of study but in every-day situations as well, it behooves us to take at least a passing glance at a few considered opinions on problem-solving; we can hope therefrom to be able to draw one or two reliable conclusions. Before undertaking a study of the function of intuition in solving problems, however, one should understand the special, directed type of thinking necessary for getting solutions. Such understanding, in turn, implies an acquaintance with the mechanism of the thinking process plus an over-all view of the ways of thinking commonly employed.

Accordingly, the present discussion will first deal
with the ways of thinking as categorized by three observers. In the first place, Cassius J. Keyser considers all varieties of thinking as falling into three main groups, the organic, the empirical, and the postulational. The simplest type, the organic, is nothing more than the response of a living organism to a stimulus. To illustrate, a one-celled animal, the amoeba, as soon as it comes into contact with an object, attempts to enclose the object, thus forming a food vacuole. Instantly, once contact has taken place, the amoeba's cytoplasm streams in such a way as to surround the object encountered. Once the fusion of the streams of cytoplasm occurs, the object, contained in a food vacuole, has become a part of the animal's structure.

Higher in the scale of thinking is the empirical type, which is experimental in nature. A certain degree of logic is involved, although not to the extent characteristic of the most complex type of thinking. An example of the empirical variety is the discovery of the formula for chemical conversion, $\frac{dx}{dt} = -kx$, in which $x$ represents the amount of unconverted substance at any particular time, $t$. Careful experimentation has revealed that the rate at which a substance is converted into another is directly proportional to the amount of unconverted substance. The right member of the formula is negative, because, as the time increases, the amount of unconverted substance decreases. As a general-

ization based entirely upon past experimentation, the formula gives a way of predicting the future behavior of substances, as regards rate of entering into combination with others.

Into the third category is placed the most complex, involved kind of thinking, the postulational. Such thinking, which is described by Keyser as deductive, is a method of reasoning that is applicable to all fields, not just to mathematics. However, since geometry lends itself so beautifully to any discussion concerning deductive reasoning, an illustration will be taken from that field. The high school sophomore, when setting out to prove the proposition that two points equally distant from the extremities of a line determine the perpendicular bisector of the line, will probably add four auxiliary lines as his first maneuver. After proving congruent two of the pairs of triangles so formed, he ultimately establishes the conclusion of the theorem.

Of course, the usual form in which the above example of deductive reasoning would be written would present a different order from that employed in more general situations, especially, the non-mathematical. That is, although deduction commonly follows the procedure from general statement to specific statement to conclusion, in the above proof the thinking would proceed from the specific statement to the conclusion and then, to the general statement. Either procedure can produce gems of precise, clear, deductive thinking.
Providing an interesting contrast to the groupings just described is the system in which four degrees of complexity are recognized. According to this second arrangement, thinking is classified under the heads of perceiving, recognizing, comparative judging, and reasoning. Listed in order of increased complexity, one notes that only the very highest type is dignified by the name reasoning.

A third method of dealing with the kinds of thinking consists of grouping all thinking under three classes, intuitive, deductive, and inductive. Such a grouping is distinctive for two reasons. For one thing, intuition is given a place among other kinds of thinking. Still more intriguing, though, is the fact that there is no attempt to label one type as simple, another, as complex. The source consulted says simply that there are three kinds of thinking and proceeds to name them.

Even a cursory glance at the three foregoing methods of classification suggests that the three men who composed the lists could not have been defining the term thinking in the same way. What is more significant in relation to the title of this chapter, however, is the inclusion in


each list of at least one kind of thinking utilized in problem solving.

In the course of classifying the ways of thinking, one inevitably speculates about the actual mechanism of thinking itself. Moreover, the mathematician is especially curious about the manner in which direction is given to his thinking. He realizes that at worst, his thinking is no more than a jumble of disconnected links, whereas, at best, it presents a chain of ideas that are neatly hooked together. He wants to analyze the situations in which his thinking has been orderly and systematized; he hopes thereby to be able to apply the all-important direction to the disconnected links, wherever and whenever he meets them.

When reviewing several cases in which his thinking has been truly fruitful, the mathematician discovers that consistently he has emphasized the structural aspects. That is, in stressing the "wholeness" of the situation, he has envisaged the given facts and sought-for solution as comprising one pattern. Success resulted from ferreting out the inter-relationships among the given facts and the ideas inferred from the given facts. However, this statement does not give the whole truth, for, just as important, if not more so, is the discovery of the relation each single fact bears to the whole pattern.

Emphasizing the whole pattern or structure has at least

two advantages over traditional logic as an explanation of the way thinking occurs. In the first place, since thinking takes place in living organisms, any explanation concerning the process of thinking must take into account the element of change. Traditional logic, however, is static; it does not allow for the changes which take place in any problem-situation during the course of a given discussion. Such changes do indeed occur, for, as each additional relationship is understood, the whole picture is seen in a new light. Furthermore, the emphasis on logic tacitly implies that an idea is simply the sum of its parts, that is, the steps leading up to the idea. In both respects, awareness of structure appears to be the more nearly accurate explanation of the mechanism of thinking, since such an explanation recognizes inevitable change and also includes the relationships of individual ideas to the whole pattern. Thus, the process of thinking appears to be characterized by a search for relationships and an appreciation of the structure of a given situation, a structure that necessarily changes from moment to moment.

At this point, the mathematician may ponder on the causes of the changes in structure experienced as he proceeds from step to step in working out a solution to a problem. Why, at a particular point, is a particular relationship suggested? What initiates his moving in a particular direction? In brief, what guides his thinking? Some may

answer that his mind is working in accordance with the prescribed rules of logic. Of course, he likes to picture himself as proceeding logically, but the question of cause still bothers him. One observer goes so far as to claim that there is almost never a completely logical discovery. Intuition is involved, nearly always; at least, intuition explains the guidance which directs the first step taken in thinking through a logical proof. Others would soften this statement considerably by asserting merely that intuition may indicate what step to take.

Inasmuch as an analytic proof to a theorem often precedes a synthetic proof, it would seem that intuition would more likely be instrumental in determining the order of the steps of the analytic proof. Thus, those who credit intuition with playing a role in the writing of proofs would probably ascribe the determination of the steps of the following proof to intuition. Lest anyone be misled, let us make clear the fact that when we refer to the determination of steps, we mean the order, not the validity of the steps. The validity, of course, is ascertained by logic.

In proving a theorem analytically, the general pattern is to proceed backwards, from the conclusion to the hypothesis. Hence, in attacking analytically the proposition that lines drawn from any point on the perpendicular bisector of a line to the extremities of the line are equal,

the student will start with the conclusion. Before he can
go through any steps, however, he needs to express the hypoth-
esis in terms of an illustration, which he either pictures
in his mind or actually draws on paper. A possible procedure
follows:
Hypothesis: \( RY \) is the perpendicular bisector of \( AB \) and
meets \( AB \) at \( Y; \) \( X \) is any point on \( RY, \) distinct from \( R \) and \( Y. \)
To prove: \( AX = BX. \)
Since corresponding parts of congruent triangles are equal,
\( AX = BX \) if 
\( \triangle AXY \cong \triangle BXY \) if

S. A. S. obtains if 
\( AY = YB \) if

\( Y \) is the midpoint of \( AB \) if 
\( Y \) fulfills the definition of midpoint if

\( XY \) is the perpendicular bisector of \( AB \) and meets
\( AB \) at \( Y. \) (Hypothesis)

\( XY \equiv XY. \)

Since all right angles are equal, \( \angle XYA = \angle XYB \) if

\( \angle XYA \) is a right angle; \( \angle XYB \) is a right angle if

\( XY \perp AB \) at \( Y \) if

\( XY \) is the perpendicular bisector of \( AB \) and meets
\( AB \) at \( Y. \) (Hypothesis)

With reference to the analytic proof outlined above,
some mathematicians would doubtless ascribe to intuition
the direction assumed by all the steps. These people credit
to intuition not only the first step, but subsequent steps
as well. Their argument is that the initial impulse to
prove a pair of triangles congruent is urged by the illustration, which reveals AX and BX to be corresponding parts of what appear to be congruent triangles. Any relationship that is suggested by a concrete figure is an appeal to intuition, they contend. Furthermore, they insist that the steps following the first are likewise suggested by intuition, for example, the step at which Y is proved to be the midpoint of AB or the one at which \( \angle XYA \) is shown to be a right angle.

A more conservative group, upon viewing the proof, would perhaps concede that only the initial direction is known intuitively. Thus this group feels that intuition is called into play only at the very beginning, when a consideration of the figure suggests the possibility of congruent triangles. Still others would point to the possibility of a combination of experience, analysis of ultimate relationship sought, and trial-and-error providing the guidance; these observers might eliminate intuition from the picture entirely. A fourth group is composed of those who, because of the sketchiness of the evidence offered by the other three groups, must remain undecided. They strongly suspect that intuition has played a part in the proof, so to speak, but cannot overlook the chance of the student's having derived direction from other sources, such as trial-and-error.

Regardless of how the order of the steps in the above proof is explained, one must bear in mind that there have
been experiences in the lives of some famous mathematicians that assuredly strengthen the case for intuition. One intriguing situation concerns a Frenchman of the seventeenth century, Pierre de Fermat. It appears that at the time of Fermat's death, someone, while going through the mathematician's papers, found a mystifying comment scribbled in a narrow margin. In that margin Fermat had indicated that he had proved the impossibility of the relation, $x^m + y^m = z^m$, when $x$, $y$, and $z$ are integers other than zero and $m$ is an integer larger than 2. However, he had lamented his not having enough space in the margin to write his proof. The mystery lies in the fact that in the three hundred years since, the proof has not been discovered by anyone, despite great efforts. The most that has been accomplished in these three centuries of work is some partial proofs. That is, proofs for some classes of values of the exponent, $m$, have been discovered. What is inexplicable is the reliance of at least one partial proof on algebraic theories that were unknown at the time of Fermat and which were not even implied remotely in his writings.

Equally puzzling is an account concerning a German geometer of a century ago, Bernhard Riemann. In his work with prime numbers Riemann emphasized a function of a


2. Ibid., 117 ff.
variable which could assume both real and imaginary values. At his death, there was found among his papers a note which stated that certain properties of the function had been deduced from an expression of the function which he had never simplified enough to publish. As of now, no one has the slightest notion of the nature of the expression.

In both cases, there appears to be some justification for the view that these men had intuitive knowledge which they either did not, or could not organize to the extent of communicating with others. Thus Fermat seems to have "sensed" a proof that, to all appearances, hinged upon relations not yet discovered. Similarly, Riemann "knew" an expression that has eluded other great thinkers down to the present. Since apparently neither mathematician could have gone through any variety of logical reasoning to reach his particular observation, each must have gained his knowledge directly, or intuitively.

Furthermore, intuition can explain the sudden insight a problem-solver may experience. In the course of solving any problem, there comes the moment when the pattern is seen with clarity; this is the moment of insight. Several examples of such insight are cited in the next chapter.

When the problem-solver is unsuccessful, that is, unable to obtain the necessary insight, he might try writing down all the steps as far as he can go. At each step he might ask himself if he is certain of the step and why he feels certain. If he comes to a step which makes him feel doubtful,
he might try to figure out what it is about the step that has shaken his confidence. Regardless of how much analyzing he does, however, he should get away from his problem for a while. Leaving the problem is advantageous for two reasons. In the first place, there is the chance that, after the rest, the student may approach his problem from an entirely new slant. For another reason, while he is engaged in something quite different from problem-solving, he just might have a flash of "intuitive insight" into his problem. If so, he will instantly see the whole pattern, and thus the solution. Some of the ways of perceiving the whole pattern will be viewed in the next chapter.
Chapter IV

Intuition and Insight

As the moment of achievement in problem-solving comes the instant that insight occurs, it would be well to examine not only the sources of insight, but also the meaning of the term. Inasmuch as some people may confuse the terms insight and intuition and may even consider them synonymous, a discussion of the relationship between the two appears warranted.

Insight means discernment, cognition, awareness of the pattern connecting ideas previously thought to be unrelated. This awareness or discernment may result from a careful analysis of the factors involved. Analysis is not always a prerequisite, however, for insight may, and often does, stem from intuition. The relationship between insight and intuition therefore, can be seen in thinking of intuition and analysis as two sources of insight.

In hopes of gaining insight by analysis, the problem-solver should continually strive to see the problem as a whole.1 He should keep viewing the problem from "above" in order to see it as a unit.

A mathematical illustration of insight stemming from analysis is drawn from the field of differential equations. A student is asked to solve the differential equation,

\[ p^4 + xp - 3y = 0, \]

in which \( y \) is the dependent variable,

x is the independent variable, and p represents the rate of change of y with respect to x. First of all, the student must be clear as to the nature of his goal. Required to find the solution to a differential equation, he must know what the word solution means when used in that particular way. At the outset then, before embarking on any steps, he visualizes his goal as a relationship involving at least one of the variables, which, together with its derivatives, satisfies the given equation. In order that the general solution in parametric form may be found, each variable must be expressed in terms of p. To express y in terms of p, it is necessary to eliminate x. The variable x can be eliminated by getting x alone in the left member and then differentiating with respect to y. (Of course, to eliminate y in a comparable manner would be just as logical a beginning.)

(1) Accordingly, since \( p^3 + xp - 3y = 0 \)

(2) \( x = \frac{3y}{p} - p^3 \)

(3) \( \frac{1}{p} = \frac{3}{p} - \left( \frac{3y}{p^2} + 3p^2 \right) \frac{dp}{dy} \)

The act of simplifying, collecting like terms, and multiplying throughout by \( \frac{1}{2p} \) dy reveals a linear equation of the first order.

(4) \( dy - \frac{3}{2} \cdot \frac{dp}{p} = \frac{3}{2} \cdot p^3 \cdot dp \)

Equation (4) is advantageous, for such an equation can be changed easily into an exact equation by applying an integrating factor, which, in this case, is \( p^{-\frac{3}{2}} \). Thus,

(5) \( p^{\frac{3}{2}} \cdot dy - \frac{3}{2} \cdot y \cdot p^{-\frac{3}{2}} \cdot dp = \frac{3}{2} \cdot p^{\frac{3}{2}} \cdot dp \)

(6) Integrating, \( y \cdot p^{\frac{3}{2}} = \frac{3}{5} \cdot p^{\frac{5}{2}} + \frac{c}{5} \)
The constant of integration is written as $c/5$ because the presence of the $3/5$ demands a multiplication by 5 in order that the equation may be cleared of fractions. Writing the arbitrary constant term originally as $c/5$ will yield $c$ for the constant after the multiplication by 5 has been carried out. Specifically, the next step is to multiply throughout by $5p^{2/5}$, yielding

$$5y = 3p^{4/5} + cp^{2/5}$$

In order to express $x$ in terms of the parameter $p$, the value for $y$ obtainable from equation (7) is substituted for $y$ in equation (2). Thus,

$$x = 3p^{-1} (\frac{3}{5} p^{4/5} + \frac{6}{5} p^{2/5}) - p^3$$

(9) Simplifying, $x = \frac{4}{5} p^{3/5} + \frac{3c}{5} p^{1/5}$

(10) Clearing of fractions, $5x = 4p^3 + 3cp^{1/5}$

The general solution consists of equations (7) and (10).

In attacking the above problem, the problem-solver's realization of the nature of the goal determined his first significant move, differentiating with respect to $y$. Next, the correct classification of equation (4) was imperative that direction for the next step be gained. The third important point consisted of the substitution performed in order to acquire equation (8).

At all three stages, insight was gained because of an ordering and classifying of the elements. At the first and third stages, direction was obtained by analyzing the type of relationship sought, an expression for each variable in terms of $p$ and one arbitrary constant. At the
second stage, an analysis of the make-up of equation (4) led to the awareness of linearity and hence to the solution of that particular equation.

In sharp contrast to the manner in which insight was gained in the preceding illustration is an interesting case concerning the German chemist, Friedrich A. Kekulé.¹ As professor of organic chemistry at Ghent University in Belgium, Kekulé had, for some time, been attempting to fathom the molecular arrangement of benzene, which arrangement appeared to elude him. One night in 1865, he had a dream so vivid that as soon as he awoke, he dashed to his desk for paper and pencil. Excitedly he drew the picture so clear in his mind. The pattern that had occurred to him while asleep was the arrangement of atoms in a molecule of benzene. The next morning he enthusiastically showed the drawing to his colleagues, who, understandably enough, were not nearly so convinced of the accuracy of the diagram as was Kekulé. Through the months, however, as experiments dealing with the behavior of benzene tended to support Kekulé's arrangement, the co-workers became proponents of his views.

¹. Lawrence Galton, "The Professor Had a Dream", Nation's Business (June, 1948), XXXVI, 67.
About eighty years after the sensational dream and about fifty years after the chemist's death, absolute verification was accomplished in the laboratories of Eastman Kodak Company. There, molecules of benzene were photographed, the photographs bearing out, in the most minute detail, the arrangement known intuitively to Kekulé years before.

Not so dramatic, but equally as thrilling is the experience of a student a year ago. Here is the problem which had teased him for two days. Let P and Q be continuous functions of x and y and have continuous derivatives, with ∂P/∂y = ∂Q/∂x, except at the points (4,0), (0,0), and (-4,0).

Let C₁ denote the circle: (x - 2)² + y² = 9; let C₂ denote the circle: (x + 2)² + y² = 9; let C₃ denote the circle: x² + y² = 25. Given that ∮₃ P dx + Q dy = 11, ∮₂ P dx + Q dy = 9, ∮₂ P dx + Q dy = 13, find ∮₄ P dx + Q dy, where C₄ is the circle: x² + y² = 1. Upon making a drawing, the student realized that C₃ contained the three circles enclosing the points of discontinuity. He knew from a previously solved problem, that under the conditions in the present problem, the line integral around C₃ was equal to the sum of the line integrals around the circles surrounding the points of discontinuity, provided, that each circle about such a point enclosed none of the other troublesome points. Herein lay the crux of the whole puzzling matter. Unfortunately, the circles C₁ and C₂ overlapped, with C₄ being internally tangent to each of the other two circles.
Perhaps we should not use the word *unfortunately* since, if the three circles had not overlapped, the exercise would have been trivial, and hence would not have posed a problem at all.

To return to the student, several times he examined the drawing, hoping to see some legal way of "moving" the circles so as to have no overlapping. He referred again and again to the hypotheses, as a check on the accuracy of his diagram. However, his circles bore the very relationships to each other that had been stipulated by the hypotheses. At last, admitting that his tactics were getting him nowhere, he laid the problem aside and went about other duties that were crying for his attention. A few hours later, while occupied with a mundane task, he had a flash of intuitive insight into the problem. All at once, while he was thinking about something utterly different, the "flash" came; right away, without going through any variety of "if-then" reasoning, he knew how to combat the overlapping circles. In this moment of insight, he saw that 
\[ \oint_C (P\,dx + Q\,dy) - \oint_C (P\,dx + Q\,dy) \] would indeed give a line integral about a circle surrounding the point \((-4,0)\), a circle that would not overlap the others. Simultaneously he saw that he could deal in a similar fashion with the point \((4,0)\). All of this he grasped in an instant, in much less time than is being required to record his experience.

The insight realized above was of the "intuitive flash" type and was not the outcome of careful analysis and
reasoning, such as characterized the solution of the differential equation. Like Kekulé, the student obtained insight into his problem when he was detached from the problem, both physically and mentally.

In all three problems described, the differential equation, the molecular arrangement of benzene and the problem dealing with line integrals, insight was the immediate fore-runner of solution. In the last two instances, penetrating insight came all at once, on an occasion hardly thought conducive to problem-solving. Insight into solving the differential equation, however, followed careful analysis and conscious application of reasoning to the problem. No claim is being made that intuition played no part at all in solving the first problem; rather, the distinction between the two methods of solution lies in the fact that on the one hand, insight was gained primarily by deliberate analysis, whereas on the other hand, insight occurred as an unexpected flash of understanding.

Of course, flashes of insight are not confined to mathematics and laboratory experiences. Such insight frequently guides actions in every-day situations as well. For instance, upon being introduced to a stranger, a person may "know" instantly that he has just found a friend. Although he may not with accuracy be able to attribute his knowledge to sudden insight into the other's character, he nevertheless senses something in the other that is in harmony with his own interests.
Then too, on the very first day of school, a teacher may need to select a pupil to run an errand. Presupposing a lack of acquaintance with her students, the teacher must rely on the kind of judgment that is not based on reasoning. She looks the class over; quickly she chooses a child she thinks is dependable. Sometimes she errs in making such "spot" estimates, but very often subsequent dealings with her pupils bear out her initial impressions.

Also, a judge will sometimes suspend sentence for no other reason than a hunch about the character of the accused; for no identifiable reason, he is confident that the other will not abuse the chance to make a new start.

Speaking of cases involving the law, several crimes have been solved because of some officer's intuitive insight. That is, the officer knows the instant he sees a particular suspect that he has found the guilty one. Because of his assurance of the suspect's guilt, he realizes that obtaining a confession is simply a matter of time. Of course, we are not referring to the type of situation in which there are tell-tale nervous mannerisms or strong circumstantial evidence. We mean the kind of case in which guilt is immediately known, even when there is not enough evidence to warrant strong suspicion.

On less serious occasions, a person sometimes will have real insight into the reliability or lack of reliability of another, but for some inexplicable reason, does not abide by his hunch. For example, a teacher is approached by a
pupil in regard to a loan. Perhaps there is something not quite convincing in the story the pupil pours out; perhaps there is something in the pupil's manner that does not ring true. Very possibly the teacher, if he will be frank with himself, is aware from the beginning of the interview of the lack of wisdom of lending the pupil money. Foolishly, the teacher reaches for his check-book and writes the pupil a check for the amount deemed necessary. True, the terms of re-payment are discussed; nevertheless, the teacher is regretting his action as soon as the pupil has left. Months later, when he is forced to admit that he will never be able to exact re-payment, he castigates himself for having closed his eyes to his first insight, unhappily correct.

Hence, insight stemming from intuition can be experienced in widely differing situations, from the arrangement of a molecule of benzene, to a problem in advanced calculus, to an acknowledgment of an introduction, to a court of law.

Concerned primarily with the crucial moment in problem-solving, the point at which insight occurs, this chapter has presented insight as a recognition of the pattern connecting the elements of a problem-situation. Furthermore, in order to see what relation insight bears to intuition, two sources of insight were examined and illustrated in some detail. Thus, the solution of the differential equation featured analysis as a preliminary to insight, whereas, in the solutions to the problems dealing with the molecular arrangement of benzene and the line integrals, as well as in the less detailed examples, intuition played a dominant role.
Chapter V

Intuition and Geometry

Now that a study of the relationship between intuition and insight has been presented, the next relationship to be regarded is that of intuition to the course so long revered in high school curricula, geometry. The subject of intuitive geometry is vigorously debated, with one faction asserting that intuition is indispensable to the study of geometry and another faction declaring with equal sincerity that intuition has no bearing whatever on the topic of geometry. Thus, with whichever faction the student aligns himself, he must be prepared to defend his position against formidable, thought-provoking attacks from the other group. Since the subject of intuitive geometry is far from shallow, it cannot be dispensed with in a summary fashion. The most we can hope to do is to present the telling arguments both for and against the intuitive approach to the study of geometry. We believe there is more evidence in favor of the one side, rather than, the other. However, we cannot forget that there are dissenters whose opinions cannot be slighted, who are equally convinced of the reasonableness of their stand.

Before proceeding further, the meaning of the phrase intuitive geometry should be clarified. Intuitive geometry, often called informal, concrete, heuristic, and experimental, is that approach to geometry which is characterized by drawing generalizations from common experiences. In the traditional
demonstrative type, on the other hand, the subject-matter is organized into a formal body of proofs. In this latter variety of geometry, each new fact is believed true by virtue of its being logically derived from statements previously accepted. In such geometry, the truth of a particular statement may or may not be immediately apparent.

An ardent defender of traditional methods and vigorous opponent to the use of intuition as a basis for studying any phase of geometry is an Austrian mathematician, Hans Hahn. Insisting upon a purely logical basis for the construction of mathematics, he points to the unreliability of intuition as evidence favoring his assertion. Thus, in support of his position, he mentions specifically two notable facts opposed to what would probably be considered "intuitively obvious". For one thing, Hahn brings out the fact that there are curves that possess no tangent at any point. Another idea equally unpalatable is his claim that there exist wave curves which cannot be generated by the motion of a point. Hahn says that offhand, anyone, when referring to experience, would think the opposite of either of these statements to be the case. By making use of convincing illustrations, though, he backs up both statements in a manner that cannot be ignored. What this mathematician

has shown, in a dramatic manner, is that intuition cannot be depended upon to guide accurately.

Hahn's chief argument against the use of intuition in geometry, however, is the point dealt with at some length in a previous chapter. Reference is being made to the fact that the word intuition is defined differently by different people. According to Hahn, the lack of definition implies the impossibility of ascribing any particular geometric truths to intuition, which, in turn, implies the utter uselessness of intuition as a basis for geometry.

At the other extreme, another mathematician, William Betz, goes so far as to assert that intuition plays an indispensable part in demonstrative geometry. With great fervor he outlines several significant supporting facts, of which a few will be presented here. In the first place, he maintains that intuition cannot be overlooked, for sooner or later, every relation must be explained by meaningful terms. Apparently, Betz thinks that, for a term to be meaningful, it must be related to experience, to ordinary observation.

Also, any use of geometric figures is an appeal to intuition. Not only are physical properties of figures

known by intuition, but positional facts as well. For instance, intersecting lines, intersecting circles, concentric circles, and adjacent angles are so understood.

Most significant, though, is the function of intuition in the so-called "logical" aspects of geometry. As indicated earlier, the validity of a particular step in a proof is designated by logic, but which step to take next is often suggested by intuition. Thus, even in the construction of a synthetic proof, some credit should go to intuition, for, previous to being built up synthetically, the proof probably was thought through analytically. During the analytic stages, intuition may have been instrumental in determining the order of the steps.

Further evidence favoring intuition in geometry is found in history. Only a novice naively credits all the relationships in Euclidean geometry to Euclid himself. Euclid's genius lay not in discovery, but rather in the ability to organize effectively relationships already known by intuition. That these relationships grasped intuitively by the people of pre-Euclidean times were not proved, did not lessen the assurance with which they were applied in practical situations. Hence, before the rise of demonstrative geometry, mathematical relationships, lacking both proof and organization, were nevertheless known to be true; such knowledge is being ascribed to intuition.

Supported by staggering evidence gathered from the necessity for meaningful terms, from the universal use of
geometric figures, from the manner in which positional facts are ascertained, from the indispensable role played in the development of an analytic proof, and from history, the case for intuition as a basis for geometry is a strong one. The question now arises as to the methods appropriate for presenting a course in intuitive geometry.

Before embarking on any kind of answer to that question, perhaps we had better make sure that no one has been misled in the previous discussion. In the references to two types of geometry, intuitive and demonstrative, we did not intend to give the impression that we are thinking of these two approaches as poles apart. That is, we have not meant to imply that a course in demonstrative geometry can make no use of intuition, or vice versa. The distinction is between emphases rather than between methods thought mutually exclusive. That the two emphases may work hand-in-hand will now be demonstrated.

First of all, in teaching beginners in demonstrative geometry, the teacher should make postulates of all facts that can be known intuitively.¹ A list of such facts follows:

1. Vertical angles are equal
2. If two straight lines are cut by a transversal so

that the corresponding angles are equal, the lines are parallel, and the converse.

3. If two straight lines are cut by a transversal so that the alternate interior angles are equal, the lines are parallel, and the converse.

4. The area of a rectangle is equal to the product of the length and the width.

5. A central angle has the same number of degrees as its intercepted arc.

6. Equal central angles have equal arcs, and the converse.

7. If, while approaching their respective limits, two variables are always equal, then their limits are equal.

8. The base angles of an isosceles triangle are equal, and other properties of figures which are evident from symmetry should be postulated.

9. Two triangles that have three pairs of corresponding sides equal are congruent, and other cases in which congruence can be determined by superposition should be postulated.

Since a student can know any of the above facts intuitively, he need not go through proofs; after all, once he sees the relationships indicated above, he is as sure as he will ever be of their acceptability. Any proof he might think through would not add a whit to his sureness.

Just as important as postulating statements intuitively acceptable is the necessity for giving emphasis to geometry
in the elementary grades. It is a lamentable fact that some schools, possibly most, stress geometry too little in the upper grades. Geometry, like foreign language, has suffered from being relegated exclusively to high school level, at least as regards most aspects of the subject. Such short-sightedness in curriculum-planning has resulted in the student's confusion, and hence, fear. All at once, when enrolled in tenth-grade geometry, the student finds himself subjected to a method of presentation different from that employed in any of his other subjects. Then too, the subject-matter of geometry clings together in a fashion new to him and utterly unlike anything previously encountered. Furthermore, he must learn a whole new vocabulary. Reflect for a little on the barrier to communication that is raised by such terms as parallel, transversal, theorem, postulate, axiom, converse, contrapositive, and perpendicular. Even the spelling of some of the terms presents hazards, as witness the word parallel. It would be interesting to compile a list of the variety of attempts at spelling this one word that may appear in only one set of final examinations!

Thus, to the student who has had no preparation for the study of geometry, the subject must seem to be a hodge-podge of facts, figures, and unfamiliar terms, which must somehow be lined up into precise, formal proofs.

To help their pupils avoid the confusion resulting from the attempt to assimilate too many new ideas too
quickly, teachers can present some of the terms and relationships earlier in the pupils' school-life. Small beginnings can even be made in the fourth and fifth grades. An interesting sideline is the fact that there actually exist textbooks in geometry that were written for the use of children as young as five years of age.¹

It seems that, at the very least, the children of the intermediate grades could be taught to identify the more common geometric figures, such as the square, the circle, and the triangle. As a special plea, let the junior high school pupil understand the true relationship between rectangles and squares, so that he will not think of those two figures as members of two distinct classes. In our own experience, we do not recall a single case of a pupil embarking upon tenth-grade geometry with the realization that a square is a special type of rectangle.

While introducing the young pupil to geometric figures, let us not limit ourselves to the plane figures. After all, the child lives in a three-dimensional world. He is familiar with boxes, tin cans, lamp shades, and ice cream cones, to name just a few of the solids that are an integral part of his life. Therefore, it is logical that he be taught to differentiate among, for example, rectangular solids,

cylinders, cones, and frustums of cones. With skillful teaching, he can be led to observe the identifying properties, both those that separate the solids and those that group various ones together.

In addition to observing the properties, the student should be required to articulate his observations, as such a requirement will train him in precise statement and clear expression. Hence, hand-in-hand with observing relationships among concrete figures, the student should be obliged to express his observations verbally, not only in order to contribute to the classwork, but especially to clarify his own thinking. Often tenth-grade students capable of making keen observations have been hampered by the inability to express themselves. Thus, training a pupil to articulate his thoughts will equip him with just as valuable an asset as will teaching him to observe relationships. Both these assets will serve him in good stead when he is launched upon demonstrative geometry.

Furthermore, the junior high school student is not too young, in our opinion, to comprehend the relativity of truth. He can surely be led to see that something is true only to the extent that something else is true, which, in turn, is true only to the extent that a third statement is acceptable. Such an understanding will give him the enormous advantage of realizing that, in any kind of reasoning, a set of basic assumptions is the starting place. We say advantage,
because, aware of the need for assumptions, he will not be surprised that his course in demonstrative geometry first lists assumptions, which he will expect to use in support of later statements.

In the junior high grades, the main goal, in regard to the parts dealing with geometry, should be to acquaint the pupils experimentally with geometric facts. Such experimentation involves intuition, for facts comprehended immediately through the senses will be accepted, without the formalization of proof.

Not only the junior high school pupils, but the tenth-grade students as well, can benefit from intuitive experimentation. It is suggested that, as much as possible, the students in high school geometry be allowed to use the intuitive approach, which utilizes constantly the students' experiences and common-sense understandings. It is recommended that there be no abrupt jump, but rather, a gradual transition, from the intuitive to the demonstrative. If well-organized and well-taught, the course in geometry can successfully combine knowledge gained without recourse to formal reasoning with knowledge derived from ideas previously accepted.

Also, by making frequent use of the analytic method of proof and of generalizations that can be drawn from inspection of concrete members of a class of figures, the teacher will call intuition into play. Best of all, in
using the analytic method of proof, the class will be able
to see how the synthetic proof is built up.

Looking back, this chapter first presented the "pros"
and "cons" of intuitive geometry. In view of the heavy
evidence favoring such course-work, it appeared worthwhile
to note some of the theorems that could very well be post-
tulated and why they could be postulated. The latter part
of the chapter dealt with suggestions for including intu-
itive geometry in junior high school arithmetic and the
advantages gained thereby. Also, reference was made to the
possible combining of the intuitive and demonstrative phases
of a tenth-grade course in geometry.
Chapter VI
Intuition and Mathematical Foundations

In the course of considering intuition as a basis for certain phases of geometry, one may naturally wonder about the role of intuition in the very foundations of mathematics. Although no clear-cut statement is possible, the positions of two of the leading schools of thought will be portrayed in hopes of shedding some much-needed light. We are referring, of course, to the intuitionist and formalist schools of mathematical foundations. Following a cursory examination of the views of the schools deemed by the writer to be the most significant, implications of opposing attitudes towards the Law of the Excluded Middle will be explored.

Probably the predominant characteristic of the formalist school, of which David Hilbert was the leading exponent, is the thought that the foundations of mathematics do not lie in logic, but rather, in prelogical symbols that are the bases for logical thinking and are understood intuitively. These prelogical symbols, the formalists argue, should be manipulated mechanically according to arbitrary rules, so that the whole body of mathematics becomes formalized into a collection of formulae.


Furthermore, the arbitrary rules followed in manipulating the symbols are restricted only to the condition that they display consistency. Such consistency, the formalists claim, is sufficient to guarantee existence of any particular number. For instance, it might be assumed that there is no such number as one with a special given property. Suppose, however, that reasoning based on a set of postulates already accepted leads to a contradiction of the assumption with regard to the number. This contradiction is enough, the formalists argue, to give assurance of the actual existence of the number. In other words, the contradiction of the assumption that there is no such number means that there is such a number. More generally, a statement is considered to be true if it can be shown to lead to no contradiction.

On the other side of the fence, the intuitionists assert that consistency does not imply existence (of a number) but rather, merely the possibility of existence. Thus, if the assumption of the existence of a particular number be contradicted, then only the possibility of the existence of the number will have been demonstrated. To be certain that a number really does exist, one must construct the number explicitly, thereby changing the possibility into an actuality.

2. Ibid., 157.
3. Ibid., 122.
4. Ibid., 157.
In addition, the intuitionists believe that knowledge of mathematical concepts is gained immediately by intuition, and not through the offices of symbolism, so revered by the formalists.

However, it is in the realm of infinite sets where the views of the two schools differ so widely. In fact, in that realm, the schools are diametrically opposed to each other. Because they limit their rules only to the condition of consistency, the formalists, in contending that consistency implies existence, necessarily hold the Law of the Excluded Middle to be applicable always, even to situations concerned with an infinite array of propositions. In order to apply the Law of the Excluded Middle to an infinite set, they surely are assuming that ordinary two-valued logic obtains. After all, if a particular proposition is believed true, the falsity of the other possibilities relative to the proposition must have been previously established. Thus the formalists are assuming that every member of the infinite set has to be either true or false.

That a member of an infinite array must be either true or false is the very point with which the intuitionists disagree, and they disagree violently. Inasmuch as not all members of an infinite set can be examined, one by one, the truth or falsity of every member cannot be ascertained.


2. According to two-valued logic, something is either true or false; hence, truth and falsity are the only possibilities.
Therefore, when dealing with infinite sets, the intuitionists maintain that a three-valued logic is requisite; that is, each member must be categorized as one of these three: true, false, or "undecidable".

Apparently, the intuitionists have a point, for undeniably, not all the members of an infinite set can be tested or scrutinized. Since not every member can be studied, it is illogical to assert that every member is either true or false. Since the application of the idea of the "excluded middle" implies just such testing, it does not appear reasonable to claim that any one point is true simply because of its being the "middle". For that matter, the words excluded middle could bear defining when used in conjunction with sets not finite. The word middle seems to indicate a group made up of a finite number of things. Hence, the use of the term in connection with an infinite number of objects is paradoxical.

On the other hand, the formalists' assertion of the truth of any statement not contradicted by an agreed-upon set of postulates is entirely acceptable to us. In any variety of logical reasoning, a set of assumptions deemed basic must be subscribed to first. Once the basic assumptions have been outlined, any proposition consistent with

the "foundation facts" is considered true. At any rate, such a proposition is true to the extent that the basic assumptions are true. Because of their emphasis on consistency, the formalists feel that if the proposition that no number with a specified property exists can be shown to be contradicted, (when reference is made to the set of postulates deemed basic), then they can be assured that the number does indeed exist.

On the matter of existence of numbers, we cannot go along with the intuitionists' insistence on the necessity of construction of the number. Perhaps they mean by construction that the number must be exhibited or demonstrated in some fashion. However, surely application of the Law of the Excluded Middle to a finite set could prove that a number with the required property exists, even though the number is not identified in any way. Of course, it could be that the intuitionists are defining existence differently than their opponents are defining the term.

In behalf of the intuitionists, though, their views regarding infinite sets are not only interesting, but seem to make sound sense as well. We are alluding especially to the thought that contradiction of the statement relative to non-existence implies only the possibility that the opposite is true. We keep thinking of how the absence
of an alibi merely indicates the possibility of the guilt of a suspect, not the actual guilt. Maybe the analogy is not too far-fetched to help bring out our point.

Throughout the course of these chapters, this paper has dealt with the role of intuition in various situations. Thus, through description and illustration, intuition has been shown to be intimately concerned not only with problem-solving, insight, and geometry, but with the ultimate foundations of mathematics as well. As with the other three topics, the relationship of intuition to foundations is a controversial one; at one pole, the formalists ascribe knowledge of symbols to intuition, and at the other, the intuitionists, with equal assurance, credit knowledge of the numbers themselves to intuition.
Epilogue

The mathematician, in accepting the challenge of a problem, may summon various aids to meet the demands involved. For one thing, he may seek ideas from people whose scholarship appears to merit confidence and who have satisfied him as to their ability to apply clear, reasonably objective thinking to questions.

As another aid, the mathematician may take mental excursions into the reasoning of various writers. Of course, these excursions entail the heavy responsibility for sorting out the ideas that appear pertinent and interpreting these ideas in his own terms. Aware of the risk that is always involved in attempting to interpret the mind of another, he can only hope that the exercise of care and discretion will effect a fair representation.

In the third place, the mathematician may, and often does examine his own experiences as teacher and student. That is, he may try to re-trace the thinking done in solving particular problems. Such introspection is very difficult, for what could be harder than to attempt to analyze one's own thinking? Since, try as he will, the mathematician cannot dissociate himself from himself, he regards the task of scrutinizing his own reasoning as his most difficult assignment. Although aware that his view of himself may be grossly inaccurate, he nevertheless forges ahead in his efforts to unravel the chain of thoughts that have led him to a solution. An individual endowed with intense curiosity,
he would probably try to study his thinking even if there were no possible "practical" outcome, such as application of the method analyzed to future problems.

Thus, when attacking a new topic that is just obscure enough to provide the necessary challenge, the mathematician may call upon opinions of his teachers, writings he deems pertinent, and his own experiences as guides.

After classifying and interpreting the ideas assembled, he may lean back in his chair with a kind of quiet elation over having found some sort of system in the ideas he has handled. Before long, however, he leans forward, reaching for paper and pencil. After all, just ahead is a new challenge.
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Vita

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Two extraordinary circumstances surrounded her undergraduate mathematical training. For one thing, because of a conflict in her schedule, she was tutored for a year by Dean H. F. Heller. Thus, all of her courses in algebra, trigonometry, and analytic geometry were studied in this way.

Another unusual circumstance lay in her taking first, the course in college geometry, which is ordinarily taken as the sixth course. This apparently unprecedented procedure was due to difficulties in scheduling classes.

She describes herself as a person interested in piano music and travel, as complements to her adventures in mathematics.