Sums of Powers and the Bernoulli Numbers

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Sums of Powers and the Bernoulli Numbers

(TITLE)

BY

Laura Elizabeth S. Coen

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

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IN THE GRADUATE SCHOOL, EASTERN ILLINOIS UNIVERSITY
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I HEREBY RECOMMEND THIS THESIS BE ACCEPTED AS FULFILLING
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ABSTRACT

This expository thesis examines the relationship between finite sums of powers and a sequence of numbers known as the Bernoulli numbers. It presents significant historical events tracing the discovery of formulas for finite sums of powers of integers, the discovery of a single formula by Jacob Bernoulli which gives the Bernoulli numbers, and important discoveries related to the Bernoulli numbers. A method of generating the sequence by means of a number theoretic recursive formula is given. Also given is an application of matrix theory to find a relation, first given by Johannes Faulhaber, between finite sums of odd powers and finite sums of even powers. An approach to finding a formula for sums of powers using integral calculus is also presented. The relation between the Bernoulli numbers and the coefficients of the Maclaurin expansion of \( f(z) = \frac{z}{e^z - 1} \), which was first given by Léonard Euler, is considered, as well as the trigonometric series expansions which are derived from the Maclaurin expansion of \( f(z) \), and the zeta function. Further areas of research relating to the topic are explored.
DEDICATION

I would like to dedicate this thesis to my son, Mark Theodore Coen, to say thanks for being a great son.
ACKNOWLEDGMENTS

I would like to thank Dr. Duane Broline for serving as my thesis committee chairperson; the committee members: Dr. Leo Comerford, Dr. Hillel Gauchman, and Dr. James Glazebrook; and Dr. Patrick Coulton for reviewing the final copy of the thesis. I also want to thank Dr. William Slough for troubleshooting TeX on my P.C., and Mrs. Sandra Reed for introducing me to TeX. Finally, I want to thank Dr. Jean-Claude Evard for suggesting the subject of sums of powers and the Bernoulli numbers.
PREFACE

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1.1 Introduction

"Of the various special kinds of numbers used in analysis, there is hardly a species that is so important and so generally applicable as the Bernoulli Numbers. Their numerous properties and applications have caused the creation of an extensive literature on the subject which still continues to attract the attention of scholars. The first statement of the properties of these numbers was given to the world by their inventor, Jacques (1) Bernoulli (1654-1705) in his posthumously printed work, *Ars Conjectandi* (Basil, 1713), pages 95 to 98." [Sm]

It is the subject of this paper to examine these Bernoulli numbers, as well as finite sums of powers to which the sequence is related. This unique number sequence has appeared in diverse areas of mathematical study and other fields, including analysis, number theory, algebraic geometry, and physics. Surprisingly, Bernoulli is known in the literature by almost as many different given names as the places where his sequence appears, including Jacques, Jacob, Jacobi, Jacobus, and James. In this thesis, we will refer to him as Jacob. Although Bernoulli was the first to give a single formula which related the coefficients appearing in the formulas for all sums of powers, mathematicians had considered formulas for sums of powers for almost 2000 years before him.

In the second chapter, we trace significant historical events which occurred in the relationship between sums of powers and the Bernoulli numbers. The formula for sums of powers of integers is often first given in introductory mathematics courses. If one considers the constant coefficients which appear in the formulas, a pattern can be discovered. A mathematician named Johannes Faulhaber who was able to identify one formula for sums of odd powers, and a second formula for sums of even powers, lived around a century before Bernoulli, and his works will be considered. After recalling Bernoulli’s contribution, we also discuss results about the sequence proved by Léonard Euler that "rank among the most elegant truths in the whole of mathematics" [Si]. We end the chapter by looking at
discoveries relating to the Bernoulli numbers in the most recent centuries.

In the third chapter, we define the Bernoulli numbers, and formulate several observations regarding properties of the sequence. We also look at two approaches to sums of powers. One approach considers Faulhaber’s original work using matrix theory, and a second allows us to extend our formulas to noninteger sums by applying integral calculus.

In the fourth chapter of this thesis, we investigate Euler’s findings on the Maclaurin expansion of \( f(z) = \frac{z}{e^z - 1} \) and its relation with the Bernoulli numbers. With this information, we will prove observations given previously. We explore the relationship between the expansion of \( f(z) \) and the expansion of several trigonometric functions. We close this chapter with a proof of a well-known result about the zeta function, \( \zeta(n) \), and compute \( \zeta(2n) \) for \( n = 1, 2, 3 \).

This thesis concludes with an exploration of areas which continue the study of sums of powers and the Bernoulli numbers in the fifth chapter.
2.1 Introduction

We begin our historical review almost 2000 years before Bernoulli’s “invention” which was discussed in Chapter 1, by considering the treatment of finite sums of powers of integers by early mathematicians from 250 B.C. to 1713 A.D. In the remainder of this chapter, we will review a small, but significant, portion of the “extensive literature on the subject” of the Bernoulli numbers from 1713 to present day.

2.2 Historical Review of Sums of Powers and the Bernoulli Numbers

One of the first developments concerning sums of powers of integers can be traced back to perhaps the greatest mechanical genius of all times, Archimedes (287–212 B.C.) [Bel]. In Syracuse around 250 B.C., he used the formula for the sums of squares, \( \sum_{n=1}^{N} n^2 \), while the first known recording of a formula for the sums of cubes was made by Āryabhaṭa I (476–?) around 500 A.D. in India [Ka].

Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham (950–1039), known in Europe as Alhazen, developed the formulas for the sums of \( k^{th} \) powers for at least \( k \) equaling one through four around the beginning of the eleventh century [Ka]. He produced a seven-book series entitled the Optics while in Egypt working as a researcher in the House of Wisdom, which used the formulas. It was his greatest contribution to mathematics, both studied and discussed in Europe for several centuries following his writing. Although ibn al-Haytham did not generalize his formulas, they can be generalized as

\[
(n + 1) \sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n} \left( \sum_{i=1}^{p} i^k \right)
\]

for positive integers \( n \) and \( k \) [Ka]. Other historical recordings in the fifteenth century Islamic world of the formula for the sum of the fourth powers, given by ibn al-Haytham, include works by the following mathematicians: Abu-ı-Hasan ibn Haydur (?–1413), and
Abu Abdallah ibn Ghazi (1437–1514), both having lived in what is now Morocco; and Ghiyath al-Din Jamshid al-Kashi (?–1429), of Samarkand (now Uzbekistan), who wrote *The Calculator’s Key* while serving in the court of Ulugh Beg as mathematician and astronomer. Katz reported that these historical recordings did not identify the manner that the formula was learned by these mathematicians, nor the purpose for which it was used by any of them.

The next significant recorded computation dealing with finite sums of powers of integers was derived by Johannes Faulhaber (1580–1635) in Ulm, Germany [Ki]. After first practicing the art of weaving as his father had before him, Faulhaber then followed a different career path by becoming a mathematician.

Faulhaber opened a school in Ulm in 1600, and from 1604 to 1610 he was salaried to run the school which became more and more an educational institution for higher mathematical sciences, with an artillery and engineering school added later [Ki]. His salary was briefly withdrawn in 1610 when Faulhaber became more concerned with physical and technical inventions and the development of an extensive library which gave less time for his teaching duties. Kirchvogel noted that it was about this time that he gave the formulas for the sums of powers for natural numbers up to the thirteenth power. Around 1615, Faulhaber derived the general formulas for the sums of powers of natural numbers [EdA2]: the sum of $n$ integers to the $r$-th power is equal to a polynomial in $n(n + 1)$ if $r$ is odd; and the sum of $n$ integers to the $r$-th power equals $(2n + 1)$ times a polynomial in $n(n + 1)$ for $r$ even. Although Faulhaber was able to find separate formulas for odd and even powers through his computations, he failed to discover a single formula for all the sums of powers which could have produced the sequence of constants now known as the Bernoulli numbers [Kn].

By 1620, the reputation of Faulhaber’s school of mathematics had spread far, and René Descartes (1596–1650) sought him out as a teacher [Ki]. Kirchvogel’s narrative of the
relationship between the two mathematicians includes a report that correspondence with
Faulhaber had been a stimulus to Descartes in writing his *Discours sum la méthode pour
bien conduire sa raison et chercher la vérité dans la sciences* in 1637. The contemporary
English reference to the text is often given as *Discourse on the method*.

Faulhaber’s lifelong passion for numbers led him to be most prolific in his computations,
perhaps more than anyone else in Europe during the first half of the 17th century [Kn].
The 1631 publication of his *Academia Algebræ* in Augsburg contained explicit formulas for
sums up to the seventeenth power, and further developed ideas by also considering sums of
sums. Faulhaber then provided an intriguing conclusion to his book with a curious exercise.
Knuth reported that the apparent intent of this ‘cryptomath’ was to prove that Faulhaber
himself had computed formulas for sums of powers to the twenty-fifth powers, inclusive.

Currently, only one original copy of *Academia Algebræ* is known to exist, located
in the collection of Cambridge University Library in England. It was reportedly once
the property of Carl Gustav Jacob Jacobi (1804–1851) who applied the Euler-Maclaurin
Summation Formula (EMSF) to sums of powers in 1834 [EdAl]. It seems the copy was in
Jacobi’s possession when he rigorously proved Faulhaber’s conjecture regarding polynomials
of alternating sign [Kn]. The EMSF (which will be discussed further in Chapter 1) was a
result of knowledge regarding sums of powers of integers [EdAl]. Although Euler attributed
the number sequence to Bernoulli (who mentioned Faulhaber in the chapter of his paper
in which the sequence was developed), it is true that Faulhaber’s work is not well known.
Edwards observed that the Faulhaber forms have been rediscovered on more than one
occasion, though they have not been correctly attributed to their original inventor.

In light of the cryptic conclusion to his book, it is also of interest to note Faulhaber’s
mystical considerations of pseudomathematical problems. He believed that he could see
"figured numbers" in certain numbers from the Bible, and he attempted to interpret future events using numbers drawn from books of the Bible including Daniel, Genesis, Jeremiah, and Revelation [Ki]. He was jailed in 1606 for predicting the end of the world by 1605, released only when he declared that his intentions were not evil in nature, but instead simply an overwhelming impulse of conscience. In 1613, he published a book which contained his attempted solution of hidden riddles of his "sealed" numbers by an unusual transposition of the German, Greek, Hebrew, and Latin alphabets for which he was censored by the clergy and the Ulm City council. Although he was successful in accurately predicting the appearance of a comet on September 1, 1618, he again received massive criticism for the publication of "secret numbers" which he had used in his calculations. Faulhaber also believed in and practiced alchemy, reporting in a correspondence on March 21, 1618, that "with the help of God, I have come to the point where I can make 2 grains of gold out of 1 grain of gold in a few days, which is why I give praise and thanks to the Almighty, and although one-tenth is supposed to become 10, up to now, I have not been able to get it any further and have worked it with my own hands" [Ki].

In his 1993 article about Faulhaber and his work with sums of powers, Donald Knuth (1938- ) published his solution to the 360-year-old secret code which Faulhaber had given in Academia Algebræ. To break the code, Knuth reported that by using a computer to denote five numerical values which Faulhaber had used to represent five letters, he found the first four corresponded to the letters IESU. Knuth believed these were part of the concealed name Jesus. The last value (which related to sums of the twenty-fourth and twenty-fifth powers) did not check out in the same way the first four values had. This led Knuth to believe that perhaps Faulhaber had not correctly computed sums of powers beyond the twenty-third power.
During the year of 1636, Pierre de Fermat (1601–1665) and Gilles Persone de Roberval (1602–1675) corresponded in France as they worked out their ideas for a formula to find the area under curves [Ka]. In October of that year, Roberval wrote to express his success in using the sums of powers of natural numbers to determine the area under curves of the form $y = x^m$. In modern notation, the area of the desired regions can be expressed by the inequalities
\[
\sum_{n=1}^{N-1} n^m < \int_0^N x^k dx < \sum_{n=1}^{N} n^m.
\]
Fermat responded to Roberval that he also knew this result and had applied the knowledge, and that he had calculated the quadrature of the "parabola" [We]. In the 1630's, Fermat had begun his work by using binomial coefficients, primarily to find theorems on sums of the form
\[
S_m(N) = \sum_{n=1}^{N} n^m
\]
or more generally $\sum_{n=1}^{N} (an + b)^m$. He employed the combinatorial identity
\[
\binom{N + M}{m + 1} = \sum_{n=1}^{N} \binom{n + m - 1}{m}
\]
from which he eventually obtained the formulas for the sums $S_m(N)$. Fermat's writings include reference to his knowledge of Archimedes's work with the sum of squares, the case of $S_3(n)$ which had been given by C.G. Bachet de Méziriac (1581–1638), and Fermat's formula for the sum of the fourth powers.

Sir Issac Newton (1642–1727) used the rule involving sums of powers (developed by Roberval, Fermat, and others in the 1630's) as he created his version of the calculus around the years 1665 to 1670 in England [Ka]. One of his main ideas was of a power series. That is, by using an area formula he was able to develop power series which he then used to derive various functions. He also used the area formula to indicate a method to obtain approximate
values for \( \sum_{n=1}^{N} \frac{1}{a+k} \) when \( N \) is large, which in this case reduced to calculating the first terms of the EMSF [Bo]. Bourbaki thus summarized that Newton and other mathematicians at the end of the seventeenth century sought solutions to calculations dealing with interpolations and the numerical evaluation of the sum of a series. This included mathematical areas such as the calculus of probabilities which considered ‘functions of large numbers’ such as the gamma function.

The mathematician who first recorded a single formula for all sums of powers of integers was Jacob I Bernoulli in Basel, Switzerland [Ho]. As a nephew was also christened with the same name, it became necessary to address him as the first of the Bernoulli’s to bear this name. Born into a family of merchants in 1654, he produced prominent results not only in mathematics, but also in mechanics and astronomy as well [Ho].

He became a professor of mathematics at the University of Basel in 1687, and his mathematical studies reached its first peak around 1689 with the beginnings of a theory of series, and the law of large numbers in probability theory [Ho]. His most original work, Ars Conjectandi (Mathematics of Probability) was given in 1704. This was his final dissertation, published eight years after his death in 1705. In the second part of this five-part work, Bernoulli dealt with the theory of combinations and introduced the numbers we now associate with his name in connection with finding sums of powers of integers. A six-part work by Ismael Bullialdus (1605–1694), Opus novum ad arithmeticum infinitorum, had been published in Paris in 1682, which recorded only the sums of the first six powers. While mentioning this in Ars Conjectandi, Bernoulli also excitedly pronounced that with the table of the sums of powers which he had computed

“... it took me less than half of a quarter of an hour to find that the tenth power of the first 1000 numbers being added together would yield the sum 91,409,924,241,424,243,424,241,924,242,500.”
This is clearly a remarkable calculation, although it is also of interest to note that it is greatly simplified by the fact that Bernoulli chose to consider the first 1000 numbers. Had he instead chosen to do the same operation for the first 982 numbers, for example, it would have been a much more difficult task.

Bernoulli’s original calculation recorded polynomial formulas for sums of powers to the tenth power, inclusive. He (as had Faulhaber before him) observed a pattern in the coefficients of the polynomials [EdA1], which we will explore in the next chapter when we consider a finite approach for deriving the Bernoulli numbers. In the second part of Ars Conjectandi, he began by first mentioning Johann Faulhaber’s work in “the contemplation of figurative numbers”. He also mentioned the works of several other scholars. Bernoulli then commented that he was unaware that a general and scientific proof of the property had been given. That is, in the mathematical works that had thus far been published on the ratios of series of squares, cubes and other powers of natural numbers, none had given an inductive argument which Bernoulli considered necessary for a scientific proof. He was one of the most significant promoters of higher analysis, placing special stress on complete induction [Ho]. Hofmann saw this as obvious in Bernoulli’s derivation of the exponential series by means of the Bernoulli numbers which he felt was a major result of the second chapter. A contrasting opinion of Bernoulli’s treatment of rigorous mathematical proof is included in Smith’s introduction of the English translation of Ars Conjectandi from Latin. Smith’s comment suggested that Bernoulli’s criticism of the lack of proof by induction previous to his writings on sums of powers was an interesting reflection on Bernoulli’s own use of incomplete induction as well.

Included in Ars Conjectandi is the poem “On Infinite Series” (translated from the Latin by Professor Helen M. Walker) which represents Bernoulli’s thoughts on the boundaries of
In this writing, Bernoulli is obviously delighted with his discoveries involving infinite series. It is clear that there is much potential in the further investigation of these series, some of which was revealed by the mathematician we will next discuss.

Léonard Euler (1707–1783) has been consistently described as one of the most prolific mathematicians in history, discovering several formulas which use Bernoulli numbers that “rank among the most elegant truths in the whole of mathematics” [Si]. In 1909, the Swiss Association for Natural Science proposed to collect and publish Euler’s scattered memoirs with financial contributions from private and mathematical society sources [Bel]. Bell conjectured in his account of the story that it must have been an unpleasant surprise that the requested funds (over $1,000,000 in present U.S. currency) would fall short! This came to light when an unknown mass of Euler’s transcripts was discovered in St. Petersburg, Russia during the collection efforts.

From 1730 to 1745 the decisive work by Euler on series and relevant questions occurred [Bo]. His desire to find the sum for

\[1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \text{etc.},\]
as Euler might have expressed it, and other similar problems gave him strong motivation to
discover a summation formula [We]. To sum the reciprocals of the squares of the integers,

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \zeta(2),$$

Euler solved a much more general problem in his investigation of the zeta function, $\zeta(n)$ [Yo].

"So much has been done on the series $\zeta(n)$", Euler wrote in 1735, "that it seems hardly
likely that anything new about them may still turn up . . . Now, however, quite unexpectedly,
I have found an elegant formula for $\zeta(2)$, depending on the quadrature of the circle [that is,
upon $\pi$]." The problem of summing the series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ was first formulated by Pietro Mengoli
(1625–1686) in 1650 [We], and had resisted solutions by other outstanding mathematicians,
including Gottfried Wilhelm Liebniz (1646–1716), Daniel Bernoulli (1700–1782), Nikolaus I
Bernoulli, (1687–1759), and James Stirling (1692–1770) [Yo]. Euler’s paper was presented
to the members of the St. Petersburg Science Academy on December 5, 1735, including
the value of $\zeta(2n)$ for $n = 2, \ldots, 6$ [We]. Georg Friedrich Bernhard Riemann (1826–1866)
was the first to extended the series to the complex numbers in his eight-page paper, *On the
number of primes less than a given magnitude*, more than one hundred years later in 1859
[EdH].

Euler’s finding continues still to be regarded as one of his most outstanding early
discoveries. The general expression for $\zeta(2n)$ was first obtained by Euler in 1735 by an
application of Newton’s formulas for the sums of powers of the roots of an equation of finite
degree [We]. He showed that

$$\zeta(2n) = r_n \pi^{2n}, \quad n \geq 1,$$

where $r_n$ are rational numbers.
In studying the zeta function, Euler sought to represent the partial sum of a series \( \sum_{n=1}^{N} f(n) \), by another infinite series involving the integral and the derivatives of the general term \( f(n) \) [Bo]. The formula discovered by Euler in 1732 [We], and independently by Colin Maclaurin (1698–1745) no later than 1738, is one of the most important in the calculus of finite differences [Yo]. It is now known as the EMSF. Maclaurin’s result was not published until 1742 in Edinburgh in his two-volume paper, *A Treatise of Fluxions* [Bo]. Although Euler’s work leading to the discovery of the EMSF began in 1730, it was not until 1755 that he realized that the coefficients which appeared in the equation were a unique sequence of numbers [Yo]. That is, in particular, that they were the sequence of numbers which it appears that Abraham de Moivre (1667–1754) was the first to designate as the “Bernoulli numbers” in his 1730 publication in London, *Miscellanea Analytica* [We]. Perhaps it was the “lack of any obvious pattern among the Bernoulli numbers” [EdA1] which caused Euler not to fully recognize for so many years the connection between the coefficients in the trigonometric series he introduced, the coefficients in the series expansion of the function, \( f(z) = \frac{z}{e^z - 1} \), and the Bernoulli numbers [Yo]. Euler was unable to use the summation formula to evaluate the series \( \zeta(n) \) for \( n = 2, 3, \ldots \), and the partial sums of the series \( \zeta(1) \), because of the rapid divergence of the absolute value of the Bernoulli numbers of even index [We]. He instead chose to sum the terms until they began “to diverge”, which enabled use of the partial sums to approximate other calculations [Yo], and to compute \( \zeta(n) \) for \( n = 2, \ldots, 16 \) with 15 or more decimal figures [We]. Siméon Denis Poisson (1781–1840) published a remainder for the EMSF in 1823 which considered the divergence of the Bernoulli numbers, thus allowing numerical computation which would evaluate the series [Gr1].

Close to a century passed before the next well-known result on Bernoulli numbers was
published. Independently discovered by Karl Georg Christian von Staudt (1798–1867) in Erlangen, Germany, and Thomas Clausen (1801–1885), the theory considered the Bernoulli numbers of even index, $b_{2n}$. In their theorems, they each described the denominator of these numbers [Bu]. Both announced their works in 1840, Staudt by publication of his paper “Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend”, and Clausen by presentation in Altona, Denmark. They proved that for $n \geq 1$,

$$b_{2n} = a_{2n} - \sum_{(p-1)|2n} 1/p,$$

where $a_{2n} \in \mathbb{Z}$ and the sum is taken over all primes $p$ where $(p - 1)|2n$ [Ir]. In particular, the result shows that there is no squared factor in the denominator of any Bernoulli number of even index [Ha].

Staudt published two later works on the theory of Bernoulli numbers in 1845 [Bu]. He evaluated the numerator of the Bernoulli numbers, $N_m$, in De numeris Bernoullianis commentatio altera. It was one of the few explicit writings ever published on the divisors of the $N_m$ [Gi]. Setting $m$ as an even integer greater than or equal to two, and $p$ a prime where $(p - 1)$ doesn’t divide $m$, Staudt proved that for an integer $r \geq 1$, if $p^r$ divides $m$, then $p^r$ also divides $N_m$. These 1845 writings were never widely known [Bu], which has been true for research regarding $N_m$ in general [Gi].

The most famous of Fermat’s recordings in the margins of his copy of Arithmetica concerned his conjecture about the diophantine equation,

$$x^p + y^p = z^p,$$

where $p > 2$ is an integer [Na]. Fermat failed to communicate the remarkable proof that he believed he had, saying only that the equation could not be satisfied for any triplet of positive integers. Almost 200 years following Fermat’s remark, Ernst Edward Kummer
(1810–1893) announced a related result while he was a professor at the University of Breslau (now Wroclaw), Poland. In 1850, Kummer showed that Fermat’s Last Theorem is true for every exponent which is a regular prime [Sh]. A regular or Kummerian prime is an odd prime \( p \) which does not divide any Bernoulli number numerator: \( b_2, b_4, b_6, \ldots, b_{(p-3)/2} \). The three irregular primes up to 100 are 37, 59 and 67. Thus, Kummer proved Fermat’s Last Theorem for every prime up to 100 except 37, 59, and 67. Nagell noted that the application of the method of infinite descent was an essential feature in Kummer’s proof. Previously, Adrien-Marie Legendre (1752–1833) and Peter Gustav Lejeune Dirichlet (1805–1859) had published solutions for \( p = 5 \) in the late 1820’s–early 1830’s [We]. Shanks remarked that the name ‘irregular’ is perhaps misleading, particularly if larger prime numbers are considered.

For \( 2 < p < 100 \), only 3 of of the first 24 odd primes are irregular. For \( 2 < p < 2520 \), 144 primes are irregular, and for the next 183 primes, \( 2520 < p < 4002 \), 72 are irregular. Analyzing these ratios gives

\[
\frac{3}{24} = .125, \quad \frac{144}{367} = .392, \quad \frac{72}{183} = .393.
\]

We can observe that the second and third ratios are substantial higher than the first.

John Couch Adams (1819–1892) was an English mathematician who shared Euler’s love for calculating exact values for mathematical constants [Gr3]. In 1877, Adams published the values of the first thirty-one Bernoulli numbers of even index. These had been calculated by hand including, for example

\[
b_{62} = \frac{-12,300,585,434,086,858,541,953,039,857,403,386,151}{6}.
\]

Grosser left unreported the method which Adams had employed in calculating the partial sequence of numbers. During his lifetime, Adams made several notable contributions to
the information about the numerators and denominators of the Bernoulli numbers of even
index.

Georgii Feodos'evich Voronoi (1868–1908) was a Russian mathematician who discov­ered, while still a student at the University of St. Petersburg in 1889, a congruence which
leads to many interesting properties of Bernoulli numbers as corollaries [Us]. Although he
was a prominent figure in Russian mathematics, many of his works have not yet been tran­slated into English [Ga]. He realized several significant mathematical results [Ba], including
a 1901 proposition on the summation of divergent series, and a 1903 work which stimulated
the development of modern analytic number theory [Wa].

Francois-Édouard-Anatole Lucas (1842–1891) is perhaps best known for devising the
mathematical puzzle, the Tower of Hanoi [Gr2]. As a professor of mathematics at both the
Lycée Saint-Louis and the Lycée Charlemagne in Paris, Lucas also loved calculating. In
1891, he published Théorie des Nombres in Paris which contained an approach to Bernoulli
polynomials by means of umbral calculus [Le]. It is possible to derive the sequence of
Bernoulli numbers from these polynomials.

Niels Nielsen (1865–1931) was a Danish mathematician who “developed no new ideas
and did not even present any fundamental theorems”, but he possessed great knowledge
and the ability to generalize existing results [Oe]. In 1923, he published the most extensive
treatise that has appeared on Bernoulli numbers, Traité Élémentaire des Nombres Bernoulli
[Ir]. Printed in Paris, the text is considered the classic source, although it has not been
translated into English to date.

An abundant amount of research has been completed on Bernoulli numbers in the
past century and continues to be generated by researchers. A 1960 publication which
surveyed research in this area was entitled On developments in an arithmetic theory of
the Bernoulli and allied numbers [Va]. The survey was followed by the 1987 publication *Bernoulli numbers: bibliography (1713–1983)* [Sk]. In 1991, the 1987 edition was updated to *Bernoulli numbers: bibliography (1713–1990)* which contained 1956 references by 839 authors of research regarding the sequence of numbers [Di]. One of the 1991 authors, Karl Dilcher, continues to collect references for future updates, and was most helpful to the author by supplying updated appendices to the 1991 edition, several interesting references, and two publications which furthered this work. The updated appendices contained almost 400 additional references which like the previous bibliographies came from France, India, China, Canada, the United States, Slovakia, Japan, Scotland, Spain, Russia, Argentina, England, Sweden, Germany, Croatia, Hungary, Singapore, Turkey, Italy, Holland, Finland, Scandinavia, and other countries. Thus, it is true that Bernoulli numbers, as well as other areas of mathematics, transcend geographical boundaries!
3.1 Introduction

This chapter addresses methods for finding finite sums of powers of integers. In the following two sections, discrete methods are used. In the final section a unique approach using a “continuous method” is given.

Section two is an expository discussion which gives a natural approach for deriving the Bernoulli numbers following a 1991 presentation by Ireland and Rosen.

Section three gathers the 1986 result of Edwards’ application of matrix algorithms to Faulhaber’s results in determining explicit formulas for finite sums of powers, with the odd and even powers treated separately.

In section four, we consider a contemporary approach to summing powers of integers by Johnson (1986) using integral calculus. This method will allow us to quickly evaluate the sum of powers by a nondiscrete method.

3.2 A Natural Approach to Deriving the Bernoulli Numbers

In A Source Book in Mathematics, Smith makes several enlightening observations regarding Jacob Bernoulli’s work on the sequence of numbers which bear his name. Smith’s thoughts continue to be true as well as relevant to the ideas in this section:

“Regardless of the fact that the discovery is more than 200 years old, mathematicians have not been able as yet to find by what process Bernoulli derived the properties of his numbers given in [Ars Conjectandi]. They can be readily derived by various modern methods, but how did he derive them with the means at his disposal?” [Sm]

A very large number of methods have been published. The interested reader can find references in [Di].

In this section, we consider a method of deriving the sequence which was presented by
Ireland and Rosen (1991) in their chapter on Bernoulli numbers. We begin by defining

\[ S_m(n - 1) = 1^m + 2^m + \cdots + (n - 1)^m, \quad \text{for all } n \in \{1, 2, \ldots\} \text{ and } m \in \{0, 1, \ldots\}. \]

It is well-known that

\[ S_0(n - 1) = (n - 1), \quad S_1(n - 1) = \frac{n(n - 1)}{2} = \frac{1}{2} n^2 - \frac{1}{2} n. \]

To obtain the Bernoulli numbers, we will establish a recursive formula for \( S_m(n - 1) \) given by the following lemma:

**Lemma.** Given \( m, n \in \mathbb{N} \), we have

\[ n^{m+1} - n = \sum_{i=1}^{m} \binom{m+1}{i} S_i(n - 1). \] (3.2.1)

**Proof:** By the binomial formula, we have

\[ (k + 1)^{m+1} = \sum_{i=0}^{m+1} \binom{m+1}{i} k^i, \quad \text{for all } k \in \{0, 1, \ldots\} \]

with the convention that \( 0^0 = 1 \). Thus

\[
\sum_{k=0}^{n-1} (k + 1)^{m+1} = \sum_{k=0}^{n-1} \sum_{i=0}^{m+1} \binom{m+1}{i} k^i = \sum_{i=0}^{m+1} \binom{m+1}{i} \sum_{k=0}^{n-1} k^i \\
= \binom{m+1}{0} \sum_{k=0}^{n-1} k^0 + \sum_{i=1}^{m+1} \binom{m+1}{i} \sum_{k=0}^{n-1} k^i \\
= n + \sum_{i=1}^{m+1} \binom{m+1}{i} S_i(n - 1).
\]

Letting \( j = k + 1 \), it follows that

\[
\sum_{j=1}^{n} j^{m+1} = n + \sum_{i=1}^{m} \binom{m+1}{i} S_i(n - 1) + S_{m+1}(n - 1).
\]
Separating terms, we get

\[
\sum_{j=1}^{n-1} j^{m+1} + n^{m+1} = n + \sum_{i=1}^{m} \binom{m+1}{i} S_i(n-1) + S_{m+1}(n-1),
\]

which yields

\[
n^{m+1} - n = \sum_{i=1}^{m} \binom{m+1}{i} S_i(n-1).
\]

Using Lemma 3.2.1, it is already possible to find \(S_2(n), S_3(n), \ldots\). For example, to get \(S_2(n)\), we apply the lemma with \(m = 2\),

\[
n^3 - n = \binom{3}{1} S_1(n-1) + \binom{3}{2} S_2(n-1).
\]

It follows by simplification of terms that

\[
S_2(n-1) = \frac{n(n-1)(2n-1)}{6} = \frac{1}{3} n^3 - \frac{1}{2} n^2 + \frac{1}{6} n.
\]

The following lemma shows that \(S_m(n-1)\) is a polynomial in \(n\).

**Lemma (3.2.2).** Let \(m \in \{1, 2, \ldots\}\), and \(n \in \{2, 3, \ldots\}\). The sums \(S_m(n-1)\) are polynomials in \(n\) of degree \(m+1\) with no constant term, and thus have the form

\[
S_m(n-1) = \sum_{j=0}^{m} c_{m,j} n^{m+1-j}
\]

where \(c_{m,j} \in \mathbb{R}\) for all \(m \in \{1, 2, \ldots\}\) and \(j \in \{0, \ldots, m\}\).

**Proof:** Since \(S_1(n-1) = \frac{1}{2} n^2 - \frac{1}{2} n\), the lemma is true for \(m = 1\). Let \(k \in \{1, 2, \ldots\}\) and assume the lemma is true for \(m \leq k\). We show the result holds when \(m = k + 1\). By Lemma 3.2.1 applied with \(m = k + 1\), we have

\[
n^{k+2} - n = \sum_{i=1}^{k+1} \binom{k+2}{i} S_i(n-1)
\]

\[
= \sum_{i=1}^{k} \binom{k+2}{i} S_i(n-1) + \binom{k+2}{k+1} S_{k+1}(n-1).
\]
By solving this last equation for $S_{k+1}(n-1)$, we see that $S_{k+1}(n-1)$ is a polynomial in $n$ of degree $k+2$ without constant term, as desired. 

Now we will develop recursive formulas to compute the coefficients $c_{m,j}$ directly. By combining formulas (3.2.1) and (3.2.2), we have:

**Lemma.** Let $c_{m,j} \in \mathbb{R}$ for all $m \in \{2,3,\ldots\}$ and $j \in \{0,1,\ldots,m\}$ be defined by

$$S_m(n-1) = \sum_{j=0}^{m} c_{m,j} n^{m+1-j}.$$  

(3.2.3)

Then, the coefficients $c_{m,j}$ satisfy the following three relationships

$$c_{m,0} = \frac{1}{m+1},$$  

(3.2.4)

$$c_{m,j} = -\frac{1}{m+1} \sum_{i=m-j}^{m-1} \binom{m+1}{i} c_{i,i+j-m}$$  

for all $j \in \{1,2,\ldots,m-1\}$, and

$$c_{m,m} = -\frac{1}{m+1} \left[ \sum_{i=1}^{m-1} \left( \binom{m+1}{i} c_{i,i} + 1 \right) \right].$$  

(3.2.6)

Further, (3.2.4), (3.2.5), and (3.2.6) define the $c_{m,j}$'s uniquely.

**Proof:** Using the relationship between formulas (3.2.1) and (3.2.2), we have

$$n^{m+1} - n = \sum_{i=1}^{m} \binom{m+1}{i} \left[ \sum_{j=0}^{i} c_{i,j} n^{i+1-j} \right]$$

$$= \sum_{i=1}^{m} \sum_{j=0}^{i} \binom{m+1}{i} c_{i,j} n^{i+1-j}.$$  

Letting $j = k - m + i$, we obtain

$$n^{m+1} - n = \sum_{i=1}^{m} \sum_{k=m-i}^{m} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k}$$

$$= \sum_{i=1}^{m} \left[ \sum_{k=m-i}^{m-1} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k} + \binom{m+1}{i} c_{i,i} n \right]$$

$$= \sum_{i=1}^{m} \binom{m+1}{i} c_{i,i} n + \sum_{i=1}^{m} \sum_{k=m-i}^{m-1} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k}. $$
We will now interchange the summation over $k$ and the summation over $i$. The above double summation is a summation over

$$A = \{(i, k) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i \leq m \text{ and } m - i \leq k \leq m - 1\}.$$

After interchanging $i$ and $k$, we obtain a summation over

$$B = \{(i, k) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq k \leq m - 1 \text{ and } m - k \leq i \leq m\}.$$

Therefore,

$$n^{m+1} - n = \sum_{i=1}^{m} \binom{m+1}{i} c_{i,i} n + \sum_{(i, k) \in A} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k}$$

$$= \sum_{i=1}^{m} \binom{m+1}{i} c_{i,i} n + \sum_{(i, k) \in B} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k}.$$

Thus, we have

$$n^{m+1} - n = \sum_{i=1}^{m} \binom{m+1}{i} c_{i,i} n + \sum_{k=0}^{m-1} \sum_{i=m-k}^{m} \binom{m+1}{i} c_{i,k-m+i} n^{m+1-k}$$

$$= \sum_{i=1}^{m} \binom{m+1}{i} c_{i,i} n + \sum_{k=1}^{m-1} \left[ \sum_{i=m-k}^{m} \binom{m+1}{i} c_{i,k-m+i} \right] n^{m+1-k}$$

$$+ \binom{m+1}{m} c_{m,0} n^{m+1}.$$

By comparing the coefficients of powers of $n$ on both sides,

$$c_{m,0} = \frac{1}{m+1},$$

$$(m+1)c_{m,k} = -\sum_{i=m-k}^{m-1} \binom{m+1}{i} c_{i,k-m+i}$$

for all $k \in \{1, \ldots, m-1\}$, and

$$(m+1)c_{m,m} = -\sum_{i=1}^{m-1} \binom{m+1}{i} c_{i,i} - 1.$$
Finally, let \( \{b_{m,j}\} \) be defined as 
\[
S_m(n-1) = \sum_{j=0}^{m} b_{m,j} n^{m+1-j}.
\]
By part 1, \( \{b_{m,j}\} \) satisfies (3.2.4), (3.2.5), and (3.2.6). Hence, \( b_{m,j} = c_{m,j} \). In particular,
\[
\sum_{j=0}^{m} c_{m,j} n^{m+1-j} = S_m(n-1).
\]

Next, computing several of the coefficients, \( c_{m,j} \), by means of the formulas (3.2.4), (3.2.5), and (3.2.6), we will look for a pattern of the coefficients as proposed in Chapter 2.

By earlier presentation, we know that
\[
c_{1,0} = \frac{1}{2}, \quad c_{1,1} = -\frac{1}{2},
\]
\[
c_{2,0} = \frac{1}{3}, \quad c_{2,1} = -\frac{1}{2}, \quad c_{2,2} = \frac{1}{6}.
\]

Applying Lemma 3.2.3 with \( m = 3 \), we have
\[
c_{3,0} = \frac{1}{4}, \quad c_{3,1} = -\frac{1}{2}, \quad c_{3,2} = \frac{1}{4}, \quad c_{3,3} = 0;
\]
and for \( m = 4 \), it gives
\[
c_{4,0} = \frac{1}{5}, \quad c_{4,1} = -\frac{1}{2}, \quad c_{4,2} = \frac{1}{3}, \quad c_{4,3} = 0, \quad c_{4,4} = -\frac{1}{30}.
\]

Computing \( c_{m,j} \) for the first values of \( m \) and \( j \), we obtain the array
\[
\begin{array}{cccc}
\frac{1}{2} & -\frac{1}{2} & \cdot \\
\frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{5} & -\frac{1}{2} & \frac{1}{3} & 0 & -\frac{1}{30} \\
\frac{1}{6} & -\frac{1}{2} & \frac{5}{12} & 0 & -\frac{1}{12} & 0 \\
\frac{1}{7} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{6} & 0 & \frac{1}{42} \\
\frac{1}{8} & -\frac{1}{2} & \frac{7}{12} & 0 & -\frac{7}{24} & 0 & \frac{1}{12} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\]
Multiplying the first row by 2, the second row by 3, \ldots, the $m^{th}$ row by $m + 1$, gives

\[
\begin{array}{cccc}
1 & -1 \\
1 & -3 & \frac{3}{2} & \frac{1}{2} \\
1 & -2 & 1 & 0 \\
1 & -3 & \frac{5}{2} & 0 & -\frac{1}{6} \\
1 & -2 & \frac{5}{2} & 0 & -\frac{1}{2} & 0 \\
1 & -\frac{7}{2} & \frac{7}{2} & 0 & -\frac{7}{6} & 0 & \frac{1}{6} \\
1 & -4 & \frac{14}{3} & 0 & -\frac{7}{3} & 0 & \frac{2}{3} & 0 \\
\end{array}
\]

\ldots

Expressing the above array by further factoring, using binomial coefficients, we have

\[
\begin{array}{c}
\binom{2}{1} (1) \binom{2}{1} (-\frac{1}{2}) \\
\binom{3}{0} (1) \binom{3}{1} (-\frac{1}{2}) \binom{3}{2} (\frac{1}{6}) \\
\binom{4}{0} (1) \binom{4}{1} (-\frac{1}{2}) \binom{4}{2} (\frac{1}{6}) \binom{4}{3} (0) \\
\binom{5}{0} (1) \binom{5}{1} (-\frac{1}{2}) \binom{5}{2} (\frac{1}{6}) \binom{5}{3} (0) \binom{5}{4} (\frac{1}{30}) \\
\binom{6}{0} (1) \binom{6}{1} (-\frac{1}{2}) \binom{6}{2} (\frac{1}{6}) \binom{6}{3} (0) \binom{6}{4} (-\frac{1}{30}) \binom{6}{5} (0) \\
\binom{7}{0} (1) \binom{7}{1} (-\frac{1}{2}) \binom{7}{2} (\frac{1}{6}) \binom{7}{3} (0) \binom{7}{4} (-\frac{1}{30}) \binom{7}{5} (0) \binom{7}{6} (\frac{1}{42}) \\
\binom{8}{0} (1) \binom{8}{1} (-\frac{1}{2}) \binom{8}{2} (\frac{1}{6}) \binom{8}{3} (0) \binom{8}{4} (-\frac{1}{30}) \binom{8}{5} (0) \binom{8}{6} (\frac{1}{42}) \binom{8}{7} (0) \\
\end{array}
\]

\ldots

We see a pattern in each column where factored numbers reveal an element common to each $j^{th}$ column. We express this in the following observation.

**Observation 1.** The coefficients $c_{m,j}$ have the form

\[
c_{m,j} = \frac{1}{m + 1} \binom{m + 1}{j} b_j, \quad (3.2.7)
\]

where $b_j$ is constant with respect to column $j$. 

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We will prove that the statement of this observation is true in Theorem 3.2.4 below.

But first, we can immediately see that if the observation is true, then the form of the coefficients \( b_1, b_2, \ldots \), is determined, after defining \( b_0 = 1 \). Indeed, (3.2.7) with \( j = m \), and (3.2.6) imply that

\[
\frac{1}{m+1} \binom{m+1}{m} b_m = - \frac{1}{m+1} \left[ \sum_{i=1}^{m-1} \binom{m+1}{i} \frac{1}{i+1} \binom{i+1}{i} b_i + 1 \right].
\]

Upon simplification, we have

\[
b_m = - \frac{1}{m+1} \sum_{i=0}^{m-1} \binom{m+1}{i} b_i.
\]

This gives the idea for the following definition of \( b_0, b_1, b_2, \ldots \), which are called the Bernoulli numbers.

**Definition**  The *Bernoulli numbers* are the rational numbers \( b_0, b_1, b_2, \ldots \), defined by the inductive formula

\[
b_0 = 1, \quad b_m = - \frac{1}{m+1} \sum_{i=0}^{m-1} \binom{m+1}{i} b_i, \quad \text{for all } m \in \{1,2,\ldots\}.
\]  

(3.2.8)

By substituting the value of \( c_{m,j} \) given by (3.2.7) into (3.2.2), we obtain the idea of the following theorem, which proves that Observation 1 is true, because the \( c_{m,j} \)'s are unique by Lemma 3.2.3. The theorem is due to [Ir]. Here we furnish a different proof.

(3.2.4) **Theorem.** The sums \( S_m \) can be directly computed by the formula

\[
S_m(n-1) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} b_k n^{m+1-k}
\]

where \( b_0, b_1, b_2, \ldots, \) are the Bernoulli numbers as defined above.

**Proof:** By Lemma 3.2.3, it is sufficient to prove that the coefficients

\[
d_{m,k} = \frac{1}{m+1} \binom{m+1}{k} b_k
\]

(3.2.9)
satisfy formulas (3.2.4), (3.2.5), and (3.2.6). Let \( m \in \{2, 3, \ldots\} \). Applying the above definition (3.2.9) and the definition of the Bernoulli numbers, we have

\[
d_{m,0} = \frac{1}{m+1} \binom{m+1}{0} b_0 = \frac{1}{m+1},
\]

so the coefficients defined by (3.2.9) satisfy formula (3.2.4). Let \( j \in \{1, 2, \ldots, m-1\} \). Let us prove that these coefficients also satisfy (3.2.5). By (3.2.9), showing the coefficients \( d_{m,k} \) satisfy (3.2.5) is equivalent to showing

\[
\frac{1}{m+1} \binom{m+1}{j} b_j = -\frac{1}{m+1} \sum_{i=m-j}^{m-1} \binom{m+1}{i} \frac{1}{i+1} \binom{i+1}{i+j-m} b_{i+j-m}.
\]

Substituting \( k = i + j - m \),

\[
\binom{m+1}{j} b_j = -\sum_{k=0}^{j-1} \binom{m+1}{k-j+m} \frac{1}{k-j+m+1} \binom{k-j+m+1}{k} b_k.
\]

After simplification, the above relation is equivalent to

\[
b_j = -\frac{1}{j+1} \sum_{k=0}^{j-1} \binom{j+1}{k} b_k,
\]

which is true by (3.2.8). Thus the coefficients \( d_{m,k} \) satisfy (3.2.5). Finally, by the definition showing the coefficients \( d_{m,k} \) satisfy (3.2.6) is equivalent to showing

\[
\frac{1}{m+1} \binom{m+1}{m} b_m = -\frac{1}{m+1} \left[ \sum_{i=1}^{m-1} \binom{m+1}{i} \frac{1}{i+1} \binom{i+1}{i} b_i + 1 \right].
\]

Simplifying, this gives

\[
b_m = -\frac{1}{m+1} \sum_{i=0}^{m-1} \binom{m+1}{i} b_i,
\]

which is true by (3.2.8).

The sequence of Bernoulli numbers of even index up to \( b_{50} \) was computed, and is located in the Appendix. We can observe the following in light of the computations:
Observation 2. The Bernoulli numbers of odd index, $b_{2n+1}$, for $n \geq 1$ are zero.

Observation 3. The Bernoulli numbers of even index, $b_{2n}$, alternate in sign.

We will prove that Observations 2 and 3 are true in Chapter 4.

3.3 Summing Powers of Integers Using Matrix Algebra

In this section, we present a novel approach applying matrix algebra to sums of powers of integers [EdA2]. To motivate our discussion, we consider the following ideas. From a computation given in the previous section, we obtain the result that $S_3(n) = \frac{n^2(n+1)^2}{4}$, and we make the striking observation that $S_3(n) = S_1(n)^2$. This raises the question whether $S_m(n)$ is a power of $S_1(n)$. It has been shown that $S_6(n) = S_1(n)^2 \left( \frac{4S_1(n) - 1}{3} \right)$, and $S_7(n) = S_1(n)^2 \left( \frac{6S_1(n)^2 - 4S_1(n) + 1}{3} \right)$ [Be2]. For $m$ odd, Beardon proved that $S_1(n)^2$ divides $S_m(n)$. If $m$ is even, he found that $S_m(n)$ is of odd degree in $n$ and is not a polynomial in $S_1(n)$. In this situation, $S_2(n)$ divides $S_m(n)$, and the quotient is a polynomial in $S_1(n)$. Beardon reported that Faulhaber knew these conclusions from his work with sums of powers.

Edwards recalled in his paper that he had learned of Faulhaber’s algorithms while following a lead from Jacob Bernoulli’s Ars Conjectandi. More precisely, Faulhaber had shown that

$$S_m(n) = S_2(n)(c_1 + c_2(S_1(n)) + c_3(S_1(n))^2 + \cdots + c_{\frac{m}{2}}(S_1(n))^{\frac{m}{2} - 1})$$

(3.3.1)

for $m$ even, and for odd $m \geq 3$, he found

$$S_m(n) = (S_1(n))^2(d_1 + d_2(S_1(n)) + d_3(S_1(n))^2 + \cdots + d_{\frac{m-1}{2}}(S_1(n))^{\frac{m-1}{2}}),$$

(3.3.2)

with $c_j$ and $d_j$ being coefficients which depend on $m$. After giving his method to obtain $S_m(n)$ for $m$ even, we obtain the result for $S_{m+1}(n)$. To enhance understanding of Faulhaber’s work, Edwards chose to present the results in matrix form for $m = 2, 3, \ldots$. We will give his results with some numerical examples computing $S_m(n)$ for specific values of $m$. 

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Edwards wrote the expansion of a telescoping series for \( k = 1, 2, \ldots, n \), using a 1654 method that was known to Blaise Pascal (1623–1662). The interested reader will find a review of Pascal's proof in [Be2]. We consider

\[
[k(k + 1)]^m - [k(k - 1)]^m = k^m \left[ \sum_{i=1}^{m} \binom{m}{i} k^{m-i} [1 - (-1)^i] \right]
\]

where \( 1 - (-1)^i \) equals 2 if \( i \) is odd, and 0 if \( i \) is even. Thus to first find the sums of odd powers, we have

\[
[k(k + 1)]^m - [k(k - 1)]^m
\]

\[
= k^m \left[ \binom{m}{1} k^{m-1} \cdot 2 + \binom{m}{3} k^{m-3} \cdot 2 + \cdots + \binom{m}{m} k^{m-m} \cdot 2 \right]
\]

\[
= 2 \left[ \binom{m}{1} k^{2m-1} + \binom{m}{3} k^{2m-3} + \cdots \right]
\]

\[
= \text{ for all } k \in \{1, 2, \ldots, n\}.
\]

Summing both sides of the equality gives

\[
\sum_{k=1}^{n} ([k(k + 1)]^m - [k(k - 1)]^m) = [n(n + 1)]^m
\]

\[
= \sum_{k=1}^{n} 2 \left[ \binom{m}{1} k^{2m-1} + \binom{m}{3} k^{2m-3} + \cdots + \binom{m}{m} k^{m+3} + \binom{m}{1} k^{m+1} \right]
\]

\[
= 2 \left[ \binom{m}{1} S_{2m-1}(n) + \binom{m}{3} S_{2m-3}(n) + \cdots + \binom{m}{m-3} S_{m+3}(n) + \binom{m}{m-1} S_{m+1}(n) \right]
\]

\[
= 2 \sum_{j=0}^{\frac{n}{2}-1} \binom{m}{2j+1} S_{2m-(2j+1)}(n).
\]

To derive the even polynomials, Edwards expanded \( k^m (k + 1)^{m+1} - k^{m+1} (k - 1)^m \), and removed the odd powers by using the expansion of \([n(n + 1)]^m\).

Applying the method specifically to \( m = 2, \ldots, 5 \), we consider the following sums:
\[
[n(n+1)]^2 = 2 \cdot \binom{2}{1} S_3(n);
\]
\[
[n(n+1)]^3 = 2 \cdot \left( \binom{3}{1} S_5(n) + \binom{3}{3} S_3(n) \right);
\]
\[
[n(n+1)]^4 = 2 \cdot \left( \binom{4}{1} S_7(n) + \binom{4}{3} S_5(n) \right);
\]
\[
[n(n+1)]^5 = 2 \cdot \left( \binom{5}{1} S_9(n) + \binom{5}{3} S_7(n) + \binom{5}{5} S_5(n) \right).
\]

Since \( S_1(n) = \frac{1}{2} n(n+1) \), expressing \( S_m(n) \) as a polynomial in \( S_1(n) \) is equivalent to expressing \( S_m(n) \) as a polynomial in \( u = n(n+1) \). As promised, we now see in matrix form the coefficients of the sums of odd powers,

\[
u^2 \cdot \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 10 & 5 \end{pmatrix} \begin{pmatrix} S_3(n) \\ S_5(n) \\ S_7(n) \\ S_9(n) \end{pmatrix},
\]

which yields

\[
\begin{pmatrix} S_3(n) \\ S_5(n) \\ S_7(n) \\ S_9(n) \end{pmatrix} = \frac{u^2}{2} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 10 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}.
\]

The coefficients of sums of even powers, in matrix form, yield

\[
\begin{pmatrix} S_2(n) \\ S_4(n) \\ S_6(n) \\ S_8(n) \end{pmatrix} = \frac{u}{2(2n+1)} \cdot \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 5 & 7 & 0 \\ 0 & 1 & 14 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}.
\]

We make several observations: Each row of the odd polynomials matrix corresponds to the rows of Pascal's triangle with every other coefficient left out; \( u^2 \) is a factor of every odd
polynomial as was suggested by formula (3.3.2); and, both matrices are in lower triangular form.

Let

\[
A = \begin{pmatrix}
2 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
0 & 4 & 4 & 0 \\
0 & 1 & 10 & 5
\end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix}
3 & 0 & 0 & 0 \\
1 & 5 & 0 & 0 \\
0 & 5 & 7 & 0 \\
0 & 1 & 14 & 9
\end{pmatrix}.
\]

Edwards gave formulas for the entries of each matrix. For \( A = (a_{i,j}) \), we have

\[
a_{i,j} = \binom{i+1}{2(i - j) + 1},
\]

and \( B = (b_{i,j}) \) gives

\[
b_{i,j} = a_{i,j} + \binom{i}{2(i - j) + 1}.
\]

Dividing the entries in each column of \( A \) by the diagonal entry in that column, and dividing the entries of each row of \( B \) by the diagonal entry in that row, we see the common elements of the two matrices

\[
A = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 4 & 4 \\
0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 5
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
3 & 0 & 0 & 0 \\
1 & 5 & 0 & 0 \\
0 & 5 & 7 & 0 \\
0 & 1 & 14 & 9
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{3} & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 7 & 0 \\
0 & 0 & 0 & 9
\end{pmatrix}
\]

Additionally, denoting

\[
C = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 5
\end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 7 & 0 \\
0 & 0 & 0 & 9
\end{pmatrix},
\]

so that \( A^{-1}C = DB^{-1} \), we have that \( A^{-1} = DB^{-1}C^{-1} \). Using Edward's method, we now see what Faulhaber saw. That is to say, by knowing only the polynomial for \( S_m(n) \), \( m \) even,
we can derive the polynomial for $S_{m+1}(n)$.

Using our computation from the previous section of this paper for $S_8(n)$, we will now compute $S_9(n)$. Edwards reported that finding $A^{-1} = DB^{-1}C^{-1}$ is equivalent to multiplication of the rows of $B^{-1}$ by $3, 5, 7, 9, \ldots$, respectively, and division of the columns by $2, 3, 4, 5, \ldots$, respectively. For the example, Edwards showed that to obtain the $j^{th}$ coefficients in the fourth row of $A^{-1}$, we must multiply the coefficients in $B^{-1}$ by 9, and divide by $(j + 1)$, $j = 1, \ldots, 4$. Given that the coefficients of the fourth row of $B^{-1}$ are

\[-\frac{1}{15} \frac{1}{5} \frac{1}{9} \frac{1}{9} \frac{2}{9} \frac{1}{9}\]

we obtain the fourth row of $A^{-1}$,

\[-\frac{3}{10} \frac{3}{5} \frac{1}{2} \frac{1}{5}\]

Thus, from

$$S_8(n) = \frac{1}{2}(2n + 1) \left( -\frac{1}{15} u + \frac{1}{5} u^2 - \frac{2}{9} u^3 + \frac{1}{9} u^4 \right)$$

we have found

$$S_9(n) = \frac{1}{2} \left( -\frac{3}{10} u^2 + \frac{3}{5} u^3 - \frac{1}{2} u^4 + \frac{1}{5} u^5 \right).$$

We see that after recalling $u = n(n + 1)$, we have

$$S_9(n) = \frac{1}{10} n^{10} - \frac{1}{2} n^9 + \frac{3}{4} n^8 - \frac{7}{10} n^6 + \frac{1}{2} n^4 - \frac{3}{20} n^2.$$

### 3.4 An Approach to Computing Sums of Powers by Integral Calculus

In this final section of Chapter 2, we consider a contemporary approach to summing the powers of the integers which was discovered by J.A. Johnson (1986). This method derives the formulas using the calculus of polynomials. We begin with the following definition which gives us a recursive relationship:
Definition  Set $f_0(x) = x$, and 

$$f_{m+1}(x) = (m + 1) \left( \int_0^x f_m(t) \, dt - x \int_0^1 f_m(t) \, dt \right) + x.$$ 

It is interesting to observe that if we would choose another function as $f_0$, we would then get some unusual functions. Next, we will prove a theorem, which was formulated from Johnson’s results, that will give us the connection between the function $f_m(n)$ and the sum $S_m(n)$.

(3.4.1) Theorem. The polynomials $f_m(n)$ satisfy the relationship

$$f_m(n) = S_m(n) = 1^m + 2^m + \cdots + n^m,$$ 

for all $n \in \{1, 2, \ldots\}$ and $m \in \{0, 1, \ldots\}$.

Proof: We first show by induction that

$$f_m(x + 1) = f_m(x) + (x + 1)^m \quad \text{and} \quad f_m(0) = 0,$$

for all $m \in \{0, 1, \ldots\}$, $x \geq 0$.

When $m = 0$, the first equation becomes $f_0(x + 1) = f_0(x) + (x + 1)^0$ for all $x \in \mathbb{R}$. We now demonstrate equality by applying the definition of $f_0$. Let $k \in \{0, 1, \ldots\}$ be such that $f_k(x + 1) = f_k(x) + (x + 1)^k$. We will prove that the equation is true when $m = k + 1$. Let $x \in \mathbb{R}$. By definition of $f_{k+1}$,

$$f_{k+1}(x + 1) - f_{k+1}(x) = (k + 1) \left( \int_0^{x+1} f_k(t) \, dt - (x + 1) \int_0^1 f_k(t) \, dt \right) + (x + 1)$$
\[-(k + 1) \left[ \int_0^x f_k(t)dt - x \int_0^1 f_k(t)dt \right] + x =
\]
\[(k + 1) \left[ \int_0^{x+1} f_k(t)dt - \int_0^x f_k(t)dt - \int_0^1 f_k(t)dt \right] + 1.\]

It follows by differentiation that

\[
\frac{d}{dx} [f_{k+1}(x + 1) - f_{k+1}(x)] = (k + 1) [f_k(x + 1) - f_k(x)].
\]

Consequently, by the induction hypothesis,

\[
\frac{d}{dx} [f_{k+1}(x + 1) - f_{k+1}(x)] = (k + 1)(x + 1)^k
\]

\[= \frac{d}{dx} (x + 1)^{k+1}.
\]

Therefore, there exists a constant \(c \in \mathbb{R}\) such that \(f_{k+1}(x + 1) - f_{k+1}(x) - (x + 1)^{k+1} = c\) for all \(x \in \mathbb{R}\). When \(x = 0\), we obtain that \(c = 0\), which proves the hypothesis for \(m = k + 1\).

Let us apply formula (3.4.1) to prove the conclusion of the theorem by induction on \(n\). Let \(m \in \{0, 1, \ldots\}\) be fixed. When \(n = 1\), by definition of \(f_m\), we have \(f_m(1) = 1\). On the other hand, we have that \(S_m(1) = 1^m = 1\). So the conclusion of the theorem is true when \(n = 1\). Let \(k \in \{1, 2, \ldots\}\) be such that \(f_m(k) = S_m(k)\). Let us prove that \(f_m(k + 1) = S_m(k + 1)\). By formula (4.4.1) applied with \(x = k\), we have \(f_m(k + 1) = f_m(k) + (k + 1)^m\).

By the hypothesis of induction, \(f_m(k) = S_m(k)\). It follows that

\[f_m(k + 1) = S_m(k) + (k + 1)^m = S_m(k + 1).
\]

Using our example from the previous section for \(S_9(n) = f_9(n)\), we will find \(S_{10}(n)\) by this new method:

\[f_{10}(x) = (9 + 1) \left[ \int_0^x f_9(t)dt - x \int_0^1 f_9(t)dt \right] + x
\]
\[= (10) \int_0^x \left( \frac{1}{10} t^{10} + \frac{1}{2} t^9 + \frac{3}{4} t^8 - \frac{7}{10} t^6 + \frac{1}{2} t^4 - \frac{3}{20} t^2 \right) dt
\]

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\[-(10x) \int_0^1 \left( \frac{1}{10} t^{10} + \frac{1}{2} t^9 + \frac{3}{4} t^8 - \frac{7}{10} t^6 + \frac{1}{2} t^4 - \frac{3}{20} t^2 \right) \, dt + x \]

\[= \frac{1}{11} x^{11} + \frac{1}{2} x^{10} + \frac{5}{6} x^9 - x^7 + x^5 - \frac{1}{2} x^3 + \frac{5}{66} x.\]

This example illustrates the efficiency of Johnson's method. We also see that our knowledge of \(S_m(n)\) is sufficient to calculate \(S_{m+1}(n)\), regardless of whether \(m\) is an even integer or not, which is an additional advantage of this technique of deriving the summation formulas. Finally, we again found that the sums are polynomials in \(n\) of degree \(m + 1\) as was proved by Lemma 3.2.2 earlier in this chapter. Yet, here we have used a continuous method to solve this discrete problem.
4.1 Introduction

The results of Chapter 4 are based on the Maclaurin expansion of the function, \( f(z) = \frac{z}{e^z - 1} \), which was first given in a 1739 publication by Euler [We]. We begin by showing a second way to obtain the recursive formula for the Bernoulli numbers given in Chapter 3, and calculate the number sequence as the Maclaurin coefficients of the Maclaurin expansion of \( f(z) \). It is also possible to begin by considering the function, \( f(x, z) = \frac{z e^{xz}}{e^z - 1} \), which generates the Bernoulli polynomials, \( B_n(x) \). As the scope of this paper is limited to the Bernoulli numbers, we will not consider the extension of the method of Bernoulli polynomials to the number sequence herein, beyond observing that \( B_n(0) = b_n \).

The power series approach yields a proof of Observation 2 from the previous chapter, that is, the Bernoulli numbers of odd index, \( b_{2k+1} \), are zero for \( k \geq 1 \). Another interesting result of the Bernoulli numbers is to provide the explicit expansion of \( \tan(z) \), \( \cot(z) \), \( \csc(z) \), and \( \coth(z) \). The Bernoulli numbers are the essential ingredient of these functions. We close this chapter with a proof of a well-known result about the zeta function, \( \zeta(n) \). Applying this result, we prove Observation 3 from Chapter 3, that is, the Bernoulli numbers of even index, \( b_{2k} \), alternate in sign. Finally we apply the result of the proof to compute \( \zeta(2n) \) for \( n = 1, 2, 3 \).

4.2 A Classical Approach to Bernoulli Numbers

We first obtain the recursive formula for the Bernoulli numbers, using the notation:

**Notation** Let \( D = \mathbb{C} \setminus \{\pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots\} \).

**Lemma.** Let \( f : D \to \mathbb{C} \) be defined by

\[
 f(0) = 1, \quad f(z) = \frac{z}{e^z - 1} \quad \text{for all } z \in D.
\]
Then $b_k = f^{(k)}(0)$ for all $k \in \{0, 1, 2, \ldots \}$, and the Maclaurin expansion of $f$ is

$$f(z) = \sum_{k=0}^{\infty} b_k \frac{z^k}{k!}, \quad |z| < 2\pi,$$

(4.2.1)

with a radius of convergence of $2\pi$.

**Proof:** The function $f$ is analytic on $D$, and the radius of convergence of the Maclaurin expansion of $f$ is $2\pi$, since $2\pi$ is the distance to the closest singular point. The interested reader is referred to [Sc] for a 1985 proof using the $n$-th root test, or a 1960 proof by the ratio test in [Mi]. Because the series converges absolutely, Formula (4.2.1) is equivalent to

$$z = (e^z - 1) \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k$$

$$= \left( \sum_{i=1}^{\infty} \frac{z^i}{i!} \right) \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k, \quad |z| < 2\pi$$

$$= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{b_k}{i!k!} z^{i+k}.$$

Let $j = i + k$. Then we have that

$$z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{b_{j-i}}{i!(j-1)!} z^j.$$

Interchanging the two sums gives

$$z = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{b_{j-i}}{j!} \left( \begin{array}{c} j \\ i \end{array} \right) z^j.$$

By comparing the coefficients of the powers in $j$, we see that (4.2.1) is equivalent to

$$b_0 = 1 \quad \text{and} \quad \sum_{i=1}^{j} \frac{b_{j-i}}{j!} \left( \begin{array}{c} j \\ i \end{array} \right) = 0, \quad \text{for all } j \in \{2, 3, \ldots \}.$$

Let $h = j - i$. Then the above relation is equivalent to

$$b_0 = 1 \quad \text{and} \quad \frac{1}{j!} \sum_{h=0}^{j-i} b_h \left( \begin{array}{c} j \\ j-h \end{array} \right) = 0.$$
Finally, by letting $m = j - 1$, we obtain that (4.2.1) is equivalent to

$$b_0 = 1 \quad \text{and} \quad \sum_{h=0}^{m} \binom{m+1}{h} b_h = 0,$$

which is true by definition (3.2.8) of the Bernoulli numbers. Thus, we have proved that

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k, \quad \text{for all } z \in \mathbb{C}, \ |z| < 2\pi.$$ 

It follows by definition of the Maclaurin expansion that $b_k = f^{(k)}(0)$, for all $k \in \{0, 1, 2, \ldots \}$.

### (4.2.2) Second Proof of Theorem (3.2.4).

Lemma (4.2.1) provides the following alternative proof of Theorem 3.2.4, which stated that

$$S_m(n - 1) = \frac{1}{m + 1} \sum_{k=0}^{m} \binom{m+1}{k} b_k n^{m+1-k},$$

using the power series (following the proof of [Ir]) as a first application of Lemma 3.2.2. For every $k \in \{0, 1, \ldots \}$, we have

$$e^{kz} = \sum_{i=0}^{\infty} \frac{(kz)^i}{i!} = \sum_{i=0}^{\infty} \frac{k^i z^i}{i!}.$$ 

Thus,

$$\sum_{k=0}^{n-1} e^{kz} = \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \frac{k^i z^i}{i!} = \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \frac{k^i z^i}{i!} = \sum_{i=0}^{\infty} \frac{z^i}{i!} S_i(n-1).$$

Applying the formula for the sum of a geometric sequence, it follows that

$$\sum_{k=0}^{n-1} (e^z)^k = (e^z)^0 + (e^z)^1 + \cdots = \frac{(e^z)^n - 1}{e^z - 1} = \frac{e^{nz} - 1}{e^z - 1}.$$
By Lemma 4.2.1,

\[
\sum_{i=0}^{\infty} \frac{z^i}{i!} S_i(n-1) = \sum_{k=0}^{n-1} e^{kz} = \frac{e^{nz} - 1}{e^z - 1}.
\]

\[
= \frac{e^{nz} - 1}{z} \cdot \frac{z}{e^z - 1} = \left( \sum_{i=0}^{\infty} \frac{(nz)^i}{i!} \right) - 1 \cdot \frac{z}{e^z - 1}.
\]

\[
= \sum_{i=1}^{\infty} \frac{n^iz^{i-1}}{i!} \sum_{k=0}^{\infty} \frac{b_k z^k}{k!}
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{n^i b_k z^{i+k-1}}{i! k!}.
\]

Let \(j = i + k - 1\). We have

\[
\sum_{i=0}^{\infty} \frac{S_i(n-1)}{i!} z^i = \sum_{i=1}^{\infty} \sum_{j=i-1}^{\infty} \frac{n^i b_{j-i+1}}{i!(j-i+1)!} z^j.
\]

Interchanging the two sums gives

\[
\sum_{j=0}^{\infty} \frac{S_j(n-1)}{j!} z^j = \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} \frac{n^i b_{j-i+1}}{i!(j-i+1)!} z^j.
\]

It follows by comparing the coefficients of \(z^j\) for every \(j \in \{0, 1, 2, \ldots\}\) that

\[
\frac{S_j(n-1)}{j!} = \sum_{i=1}^{j+1} \frac{n^i b_{j-i+1}}{i!(j-i+1)!}.
\]

Let \(m = j\) and \(k = m - i + 1\). Following simplification, we obtain the result

\[
S_m(n-1) = \frac{1}{(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} b_k n^{m+1-k}, \quad \text{for all } m \in \{1, 2, \ldots\}.
\]

By Lemma 4.2.1, it was shown that \(b_k = f^{(k)}(0)\) for all \(k \in \{0, 1, \ldots\}\) where \(f^{(0)}(0) = 1 = b_0\) and

\[
f(z) = \frac{z}{e^z - 1}, \quad \text{for all } z \in \mathcal{D}.
\]

By taking the derivatives of the function and evaluating at \(z = 0\) (since we know \(f\) is analytic at the point 0), we can compute the Bernoulli numbers. It can be seen that the
method quickly becomes very involved. To get $b_1$, we need $f'(z)$:

$$f'(z) = \frac{(e^z - 1) - ze^z}{(e^z - 1)^2}, \quad \text{and} \quad f'(0) = \lim_{z \to 0} f'(z).$$

Then using the series expansion for $e^z$,

$$f'(0) = \lim_{z \to 0} \frac{\left[\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right) - 1\right] - z \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right)}{\left(z \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}\right)^2}.$$

This reduces to

$$f'(0) = \lim_{z \to 0} \frac{-\frac{1}{2} + \left[\left(\frac{z}{2!} + \cdots\right) - \left(\frac{z}{1!} + \cdots\right)\right]}{(1 + \frac{z}{2!} + \frac{z^3}{3!} + \cdots)^2} = -\frac{1}{2} = b_1.$$

Alternately, by long division we can determine the Bernoulli numbers, again using the series expansion for $e^z$. Since

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots,$$

then,

$$z \div \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right) = 1 - \frac{1}{2} \cdot z + \frac{1}{6} \cdot \frac{z^2}{2!} + 0 + \cdots = b_0 + b_1 z + \frac{b_2 z^2}{2!} + \frac{b_3 z^3}{3!} + \cdots,$$

where $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, and $b_3 = 0$. This method appears more efficient for calculating values when compared to the previous method.

Lastly in this section, we prove Observation 3 from Chapter 3 using the function. We begin by considering the following lemma:

\[ (4.2.3) \textbf{Lemma.} \quad \text{If } h(z) = \sum_{k=0}^{\infty} c_k z^k \text{ is analytic on a domain } D_h \text{ and is even, then } c_{2k+1} = 0 \text{ for all } k \in \{0, 1, \ldots\}. \]
**Proof:** Let \( z \in D_h \). By the definition of an even function, we have

\[
0 = h(z) - h(-z) = \sum c_k z^k - \sum c_k (-z)^k \\
= \sum c_k [z^k - (-z)^k] \\
= \sum c_k [1 - (-1)^k] z^k \\
= \sum 2c_{k+1} z^{2k+1}.
\]

This implies that \( c_{2k+1} = 0 \) for all \( k \in \{0, 1, 2, \ldots \} \) by uniqueness of the Maclaurin expansion.

This leads us to the proposition which will give us the desired result:

(4.2.4) **Proposition.** The Bernoulli numbers of odd index, \( b_{2k+1} \), are zero for all \( k \in \{1, 2, \ldots \} \).

**Proof:** Let \( z \in D \). Let \( g(z) = \frac{z}{e^z - 1} + \frac{z}{2} \). By Lemma 4.2.1, we have

\[
g(z) = 1 + \sum_{k=2}^{\infty} \frac{b_k z^k}{k!}.
\]

Let us check that \( g(z) \) is an even function:

\[
g(z) = \frac{2z + z(e^z - 1)}{2(e^z - 1)} = \frac{z(1 + e^z)}{2(e^z - 1)}.
\]

On the other hand,

\[
g(-z) = \frac{-z(1 + e^{-z})}{2(e^{-z} - 1)} \cdot \frac{e^z}{e^z} = \frac{-z(1 + e^z)}{2(1 - e^z)} = g(z).
\]

Therefore by Lemma 4.2.3, we have \( b_{2k+1} = 0 \) for all \( k \in \{1, 2, \ldots \} \).
4.3 Trigonometric Series Expansions in Terms of Bernoulli Numbers

We will now apply \( f(z) = \frac{z}{e^z - 1} \) to obtain trigonometric series expansion for \( \tan(z) \), \( \cot(z) \), \( \csc(z) \), and \( \coth(z) \). As previously noted, works by Euler first connected the trigonometric functions and their series and product expansions with the Bernoulli numbers. Since then, many interesting results have been scattered throughout the mathematical literature [De]. Recent publications which have considered the topic include [De], [Gr1], and [Si].

We will use the following notation:

**Notation** Given \( a \in \mathbb{C} \) and \( r \in (0, \infty) \), we denote by \( D(a, r) \) the open disk with center \( a \) and radius \( r \), that is, \( D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \).

Although the hyperbolic cotangent is not always of immediate interest, we consider it first because of the ease which with we can find it using a power series. This result will then lead us to other trigonometric series expansions. Thus, we arrive at our first proposition:

**4.3.1 Proposition.** We have

\[
\frac{z}{e^z - 1} = \frac{z}{2} \left( \coth \frac{z}{2} - 1 \right), \quad \text{for all } z \in D(0, 2\pi).
\]

**Proof:** The left-hand side of our equation has been shown to be analytic on \( D \). Let \( z \in D(0, 2\pi) \). Then \( \frac{z}{2} \coth \frac{z}{2} = \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} \) by applying the definition of hyperbolic cotangent. Simplifying, we get

\[
\frac{z}{2} \cdot \frac{e^{-z/2} e^z + 1}{e^{-z/2} e^z - 1} = \frac{z}{2} \left( \frac{e^z - 1}{e^z - 1} + \frac{2}{e^z - 1} \right) = \frac{z}{2} + \frac{z}{e^z - 1}.
\]

Now we can show our first connection to the Bernoulli numbers.
(4.3.2) Proposition. **The power series expansion of** \( z \coth z \) **is**

\[
z \coth z = \sum_{j=0}^{\infty} 4^j b_{2j} \frac{z^{2j}}{(2j)!}, \quad \text{for all} \ z \in D(0, \pi).
\]

**Proof:** Let \( z \in D(0, \pi) \). By substituting \( z \) for \( \frac{z}{2} \) in Proposition 4.3.1, we find

\[
\frac{2z}{e^{2z} - 1} = z(\coth z - 1)
\]

hence,

\[
z \coth z = \frac{2z}{e^{2z} - 1} + z.
\]

By substituting \( 2z \) for \( z \) in Theorem 4.2.1, we obtain

\[
z \coth z = \sum_{k=0}^{\infty} \frac{b_k}{k!} (2z)^k + z.
\]

It follows by Proposition 4.2.4, which proved that \( b_3 = b_5 = b_7 = \ldots = 0 \), with \( k = 2j \),

\[
z \coth z = \sum_{j=0}^{\infty} \frac{b_{2j}}{(2j)!} (2z)^{2j} + \frac{b_1}{1!} (2z)^1 + z
\]

\[
= \sum_{j=0}^{\infty} \frac{b_{2j}}{(2j)!} (2z)^{2j} - z + z
\]

\[
= \sum_{j=0}^{\infty} 4^j b_{2j} \frac{z^{2j}}{(2j)!}.
\]

This result leads us to the connection between the cotangent and hyperbolic cotangent functions which we present as our next proposition.

(4.3.3) Lemma. **The relationship** \( \cot z = i \coth(iz) \) **is true for all** \( z \in D(0, \pi) \).

**Proof:** Let \( z \in D(0, \pi) \). Using the identities

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \text{and} \quad \cos z = \frac{e^{-iz} + e^{iz}}{2}, \quad \text{for all} \ z \in \mathbb{C},
\]

we find that

\[
\cot z = \frac{\cos z}{\sin z} = \frac{1}{i} \cdot \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = \frac{i}{e^{iz} - e^{-iz}} = i \coth(iz).
\]
Next, we connect the cotangent function to the Bernoulli numbers:

(4.3.4) **Proposition.** The power series expansion for \( \cot z \) is

\[
\cot z = \sum_{j=0}^{\infty} (-4)^j b_{2j} \frac{z^{2j}}{(2j)!} \quad \text{for all } z \in D(0, \pi).
\]

**Proof:** Let \( z \in D(0, \pi) \). By Lemma 4.3.3, we have \( \cot z = i \coth(iz) \). Consequently, it follows by substituting \( iz \) for \( z \) in Proposition 4.3.2 and using the fact that \(|iz| = |z|\),

\[
z \cot z = (iz) \coth(iz) = \sum_{j=0}^{\infty} 4^j b_{2j} \frac{(iz)^{2j}}{(2j)!}
= \sum_{j=0}^{\infty} 4^j b_{2j} \frac{(-1)^j z^{2j}}{(2j)!}.
\]

To find the relationship between the tangent and cotangent functions, we establish the next lemma.

(4.3.5) **Lemma.** The equality \( \tan z = \cot z - 2 \cot(2z) \) holds for all \( z \in D(0, \frac{\pi}{2}) \).

**Proof:** Let \( z \in D(0, \frac{\pi}{2}) \). We obtain the conclusion by applying the formula given in Lemma (4.3.3) for \( \cot z \) for all \( z \in \mathbb{C} \) such that \( \Re(z) \neq k\pi \) with \( k \in \mathbb{Z} \), and for \( \cot(2z) \), \( \Re(2z) = 2\Re(z) \neq k\pi \), that is \( \Re(z) \neq k \cdot \frac{\pi}{2} \).

This enables us to establish our third relationship between a trigonometric function, the tangent, and the Bernoulli numbers.
(4.3.6) Proposition. The power series expansion of \( \tan z \) is

\[
\tan z = \sum_{j=1}^{\infty} (-1)^{j+1} 4^j (4^j - 1) b_{2j} \frac{z^{2j-1}}{(2j)!}, \quad \text{for all } z \in D(0, \frac{\pi}{2}).
\]

Proof: Let \( z \in D(0, \frac{\pi}{2}) \). By Lemma 4.3.5, \( \tan z = \cot z - 2 \cot(2z) \). Hence, by Proposition 4.3.4,

\[
z \tan z = \sum_{j=0}^{\infty} \frac{(-4)^j b_{2j}}{(2j)!} z^{2j} - \sum_{j=0}^{\infty} \frac{(2z)^{2j}}{(2j)!}
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} b_{2j} 4^j (4^j - 1) z^{2j}}{(2j)!}
\]

and the conclusion follows since \( \tan 0 = 0 \).

To identify the final relationship we will consider in this section between trigonometric functions and Bernoulli numbers, we need some additional identities which we give as the following lemma.

(4.3.7) Lemma. The equality \( \csc z = \cot z + \tan \left( \frac{\pi}{2} \right) \) is true for all \( z \in D(0, \pi) \).

Proof: Let \( z \in D(0, \pi) \). Using the identity \( \csc z = \frac{2i}{e^{iz} - e^{-iz}} \), by Lemma 4.3.3 the identity for \( \cot z \), and the result of Lemma 4.3.5 for \( \tan z \), we establish the equality.

Our final proposition will show the relationship between the cosecant function and the sequence of Bernoulli numbers.
(4.3.8) **Proposition.** The power series expansion of \( \csc z \) is

\[
z \csc z = \sum_{j=0}^{\infty} (-1)^{j+1} 2(2^{2j-1} - 1)b_{2j} \frac{z^{2j}}{(2j)!}, \quad \text{for all } z \in D(0, \pi).
\]

**Proof:** Let \( z \in D(0, \pi) \). By Lemma 4.4.7,

\[
z \csc z = z \cot z + z \tan \left( \frac{z}{2} \right)
= \sum_{j=0}^{\infty} 4^j b_{2j} \frac{(-1)^j z^{2j}}{(2j)!} + z \sum_{j=0}^{\infty} (-1)^{j+1} 4^j (4^j - 1)b_{2j} \frac{\left( \frac{z}{2} \right)^{2j}}{(2j)!}
= \sum_{j=0}^{\infty} \frac{b_{2j} 4^j (-1)^j}{(2j)!} \left[ 1 - \frac{4^j - 1}{2^{2j-1}} \right] z^{2j}
= \sum_{j=0}^{\infty} \frac{b_{2j} 4^j (-1)^j}{(2j)!} \cdot \frac{4^j \cdot \frac{1}{2} - 4^j + 1}{4^j \cdot \frac{1}{2}} z^{2j}
= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} b_{2j} \cdot 2(-2^{2j-1} + 1) z^{2j}
= \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j)!} b_{2j} \cdot 2(2^{2j-1} - 1) z^{2j}
= \sum_{j=0}^{\infty} (-1)^{j+1} 2(2^{2j-1} - 1)b_{2j} \frac{z^{2j}}{(2j)!}.
\]

Thus, we have explicitly shown the series expansion for \( \coth z \), \( \cot z \), \( \tan z \), and \( \csc z \), in terms of the Bernoulli numbers. This is an efficient way of dealing with a trigonometric series expansion which if expanded in a standard manner can quickly produce expressions which are cumbersome. The following example will emphasize exactly that point. Consider the series expansion of \( \csc(z) \):

\[
\csc z \approx \frac{1}{2} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15,120}.
\]
The denominator in the subsequent terms has grown at an incredible rate which makes considering additional terms in a traditional manner involved, and not easily predicted. By applying our final proposition,

\[
\csc z \approx \sum_{j=0}^{3} (-1)^{j+1} 2^{2j-1} (2^{2j-1} - 1) b_{2j} \frac{z^{2j-1}}{(2j)!}
\]

\[= \left[ (-1)^2 \left( -\frac{1}{2} \right) \frac{z^{-1}}{0!} \right] + \left[ (1) 2 \left( \frac{1}{6} \right) \frac{z}{2} \right] + \left[ (-1)^2 (7) \left( -\frac{1}{30} \right) \frac{z^3}{24} \right] + \left[ (1) 2 (31) \left( \frac{1}{42} \right) \frac{z^5}{720} \right]
\]

\[= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15,120}.\]

As a consequence of our earlier computations, we will now show the relationship between the zeta function and the Bernoulli numbers. Let us recall the definition of the Riemann zeta function.

**Definition** The Riemann zeta function, \( \zeta(z) \), is defined by

\[
\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad \text{for all } z \in \mathbb{C}, \Re(z) > 1,
\]

recalling that \( k^z = e^{z \ln k} \).

Now we will consider \( \zeta(2j), j = 1, 2, 3, \ldots \) in our final theorem of this chapter, where we follow the proof of [Sc].

**4.3.9 Theorem.** For every \( j \in \{1, 2, 3, \ldots\} \),

\[
\zeta(2j) = \frac{(-1)^{j+1} 2^{2j-1} \pi^{2j}}{(2j)!} b_{2j}.
\]
Let $z \in D(0, 2\pi)$, and $x = \frac{z}{\pi}$. Then

$$z \cot z = \pi x \cot(\pi x) = 1 + x \sum_{n=1}^{\infty} \left( \frac{1}{x+n} + \frac{1}{x-n} \right)$$

by the method of partial fraction expansion of the cotangent. The interested reader can find several approaches to deriving this expansion in [Ir], [Ko], and [Si]. Scharlau gives the following proof which leads us to the zeta function as

$$z \cot(z) = 1 + \frac{z}{\pi} \sum_{n=1}^{\infty} \left( \frac{1}{\frac{z}{\pi} + n} + \frac{1}{\frac{z}{\pi} - n} \right)$$

$$= 1 + z \sum_{n=1}^{\infty} \left( \frac{1}{z + n\pi} + \frac{1}{z - n\pi} \right)$$

$$= 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2} \left( \frac{1}{1 - z^2/n^2\pi^2} \right)$$

$$= 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2} \sum_{k=0}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}}$$

$$= 1 - 2 \sum_{k=0}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \right) \frac{z^{2k+2}}{\pi^{2k+2}}.$$  

Letting $j = k + 1$, we have

$$z \cot(z) = 1 - 2 \sum_{j=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2j}} \right) \frac{z^{2j}}{\pi^{2j}}.$$  

By uniqueness of the Maclaurin expansion of $z \cot z$, it follows that

$$1 - 2 \sum_{j=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2j}} \right) \frac{z^{2j}}{\pi^{2j}} = 1 + \sum_{j=1}^{\infty} (-4)^j b_{2j} \frac{z^{2j}}{(2j)!}$$

$$= 1 + \sum_{j=1}^{\infty} \frac{2^{2j}(-1)^j b_{2j} z^{2j}}{(2j)!},$$

yielding

$$\sum_{n=1}^{\infty} \frac{1}{n^{2j}} = \frac{2^{2j-1} \pi^{2j} (-1)^{j+1}}{(2j)!} b_{2j},$$

which we know as the zeta function for even integers, $\zeta(2n)$.  

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which we know as the zeta function for even integers, \( \zeta(2n) \).

Additionally, we have proof of Observation 3 from Chapter 3 in the following corollary to Theorem 4.3.9.

**Corollary.** The Bernoulli numbers of even index, \( b_{2j} \), alternate in sign.

**Proof:** It is obvious that \( \sum_{n=1}^{\infty} \frac{1}{n^{2j}} \) is a positive series. Thus it is a consequence of Theorem 4.3.9 that the \( b_{2j} \) have alternating sign.

**Corollary.** By Theorem 4.3.9, we have

\[
\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^2}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.
\]

**Proof:**

\[
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2 \cdot \pi^2 \cdot (-1)^2 \cdot \frac{1}{6}}{2} = \frac{\pi^2}{6},
\]

\[
\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8 \cdot \pi^4 \cdot (-1)^3 \cdot (-\frac{1}{3!})}{4!} = \frac{\pi^4}{90},
\]

\[
\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{32 \cdot \pi^6 \cdot (-1)^4 \cdot \frac{1}{4!}}{720} = \frac{\pi^6}{945}.
\]
5.1 Conclusion

The objective of this thesis was to explore the relationship between sums of powers and the Bernoulli numbers. The subject of the Bernoulli numbers is very deep, and can lead into many areas beyond the number sequence. In these concluding remarks, it seems appropriate to discuss areas of further study.

By concentrating on the number sequence alone, one can connect ideas in many areas. The Bernoulli polynomials, from which the Bernoulli numbers can be found, are an additional area of study. Another possibility is to study the Euler numbers and Euler polynomials, which can be derived from the Bernoulli numbers. Further study of the zeta function, which was briefly considered in the previous chapter, and the Euler-Maclaurin Summation Formula, which was discussed historically in Chapter 2, are additional areas which could lead to interesting results.
REFERENCES


[Sk] L. Skula, and I. Slavutskii, Bernoulli numbers bibliography (1713-1983), J.E. Purkyne
University, Brno, (1987).


Appendix

This appendix contains a list of the Bernoulli numbers, \( b_{1} \) to \( b_{50} \), and the decimal approximation.

\[
\begin{align*}
1, & \quad \frac{1}{2}, \quad -0.5000000000 \\
2, & \quad \frac{1}{6}, \quad 0.1666666667 \\
& \quad 3, \quad 0, \quad 0 \\
4, & \quad \frac{1}{30}, \quad -0.03333333333 \\
& \quad 5, \quad 0, \quad 0 \\
6, & \quad \frac{1}{42}, \quad 0.02380952381 \\
& \quad 7, \quad 0, \quad 0 \\
8, & \quad \frac{1}{30}, \quad -0.03333333333 \\
& \quad 9, \quad 0, \quad 0 \\
10, & \quad \frac{5}{66}, \quad 0.07575757576 \\
& \quad 11, \quad 0, \quad 0 \\
12, & \quad \frac{-691}{2730}, \quad -0.2531135531 \\
& \quad 13, \quad 0, \quad 0 \\
14, & \quad \frac{7}{6}, \quad 1.1666666667 \\
& \quad 15, \quad 0, \quad 0 \\
16, & \quad \frac{-3617}{510}, \quad -7.092156863 \\
& \quad 17, \quad 0, \quad 0 \\
18, & \quad \frac{43867}{798}, \quad 54.97117794 \\
& \quad 19, \quad 0, \quad 0 \\
20, & \quad \frac{-174611}{330}, \quad -529.1242424
\end{align*}
\]
21, 0, 0
22, \( \frac{854513}{138} \), 6192.123188
23, 0, 0
24, \( \frac{-236364091}{2730} \), -86580.25311
25, 0, 0
26, \( \frac{8553103}{6} \), .1425517167 \( 10^7 \)
27, 0, 0
28, \( \frac{-23749461029}{870} \), -.2729823107 \( 10^8 \)
29, 0, 0
30, \( \frac{8615841276005}{14322} \), .6015808739 \( 10^9 \)
31, 0, 0
32, \( \frac{-7709321041217}{510} \), -.1511631577 \( 10^{11} \)
33, 0, 0
34, \( \frac{2577687858367}{6} \), .4296146431 \( 10^{12} \)
35, 0, 0
36, \( \frac{-26315271553053477373}{1919190} \), -.1371165521 \( 10^{14} \)
37, 0, 0
38, \( \frac{2929993913841559}{6} \), .4883323190 \( 10^{15} \)
39, 0, 0
40, \( \frac{-261082718496449122051}{13530} \), -.1929657934 \( 10^{17} \)
41, 0, 0
42, \( \frac{1520097643918070802691}{1806} \), .8416930476 \( 10^{18} \)
43, 0, 0
44, \( \frac{-27833269579301024235023}{690} \), -.4033807185 \( 10^{20} \)
45, 0, 0
46, $\frac{596451111593912163277961}{282}$, $2.115074864 \times 10^{22}$

47, 0, 0

48, $\frac{-5609403368997817686249127547}{46410}$, $-1.208662652 \times 10^{24}$

49, 0, 0

50, $\frac{495057205241079648212477525}{66}$, $0.7500866746 \times 10^{25}$