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Leo Comerford
Eastern Illinois University, lpcomerford@eiu.edu

Charles C. Edmunds
MOUNT ST. VINCENT UNIVERSITY

Gerhard Rosenberger
UNIVERSITXT DORTMUND

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COMMUTATORS AS POWERS IN FREE PRODUCTS OF GROUPS

LEO P. COMERFORD, JR., CHARLES C. EDMUNDS, AND GERHARD ROSENBERGER

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ABSTRACT. The ways in which a nontrivial commutator can be a proper power in a free product of groups are identified.

It is well known that in a free group, a nontrivial commutator cannot be a proper power. This seems to have been noted first by Schützenberger [2]. It is, however, possible for a nontrivial commutator to be a proper power in a free product. Our aim in this paper is to determine the ways in which this can happen.

Theorem 1. *Let* $G = *_{i\in I} G_i$, *the free product of nontrivial free factors* G_i. *If* $V, X, Y \in G$ *and* $V^m = X^{-1}Y^{-1}XY = [X, Y]$ *for some* $m \geq 2$, *then either*

(1.1) $V \in W^{-1}G_iW$ *for some* $W \in G$, $i \in I$, *and* V^m *is a commutator in* $W^{-1}G_iW$; *or*

(1.2) m *is even,* $V = AB$ *with* $A^2 = B^2 = 1$, *and* $V^m = [A, B(AB)^{(m-2)/2}]$; *or*

(1.3) m *is odd,* $V = AC^{-1}AC$ *with* $A^2 = 1$, *and* $V^m = [A, C(AC^{-1}AC)^{(m-1)/2}]$; *or*

(1.4) $m = 6$, $V = AB$ *with* $A^2 = B^3 = 1$, *and* $V^6 = [B^{-1}ABA, B(AB)^2]$; *or*

(1.5) $m = 3$, $V = AB$ *with* $A^3 = B^3 = 1$, *and* $V^3 = [BA^{-1}, BAB]$; *or*

(1.6) $m = 2$, $V = AB$ *with* $A^2 = 1$ *and* $B^{-1} = C^{-1}BC$ *for some* $C \in G$, *and* $V^2 = [C^{-1}A, B]$; *or*

(1.7) $m = 4$, $V^2 = ABC$ *with* $A^2 = B^2 = C^2 = 1$, *and* $V^4 = [BA, BC]$.

We recall that in a free product every element of finite order lies in a conjugate of a free factor. Thus we have the following consequence of Theorem 1.

Corollary 2. *Let* $G = *_{i\in I} G_i$, *where no* G_i *has elements of even order. If* $V, X, Y \in G$ *and* $V^m = [X, Y]$ *for some* $m \geq 2$, *then either* $V \in W^{-1}G_iW$ *for some* $W \in G$, $i \in I$, *and* V^m *is a commutator in* $W^{-1}G_iW$ *or* $m = 3$, $V = AB$ *for some* $A, B \in G$ *with* $A^3 = B^3 = 1$, *and* $V^3 = [BA^2, BAB]$.

Part (1.7) of Theorem 1 is somewhat unsatisfactory in that it describes the form of V^2 rather than that of V. Among the ways in which an element V of a free product may have $V^2 = ABC$ with $A^2 = B^2 = C^2 = 1$ is $V = DE$

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with $D^2 = E^4 = 1$, in which case $V^2 = (D)(E^2)(E^{-1}DE)$. Not every solution is of this form, as shown by $G = \langle a, b\ ;\ a^2 = b^2 = (ab)^2 = 1\rangle * \langle c\ ;\ c^2 = 1\rangle$ and $V = acbcabc$; here $V^2 = (acbca)(bcacb)(cabc)$, a product of three elements of order two, but V is not a product of two elements of finite order. A classification of elements V satisfying the conditions of (1.7) has eluded us.

Relative to (1.6), we record the following well-known consequence of the Conjugacy Theorem for Free Products [1, Theorem IV.1.4].

Lemma 3. *If B is an element of a free product $G = *_{i\in I}G_i$ and $B^{-1} = C^{-1}BC$ for some $C \in G$, then either*

(3.1) *$B \in W^{-1}G_iW$ for some $W \in G$, $i \in I$, and there is a $C \in W^{-1}G_iW$ such that $B^{-1} = C^{-1}BC$ or*

(3.2) *$B = DE$ for some $D, E \in G$ with $D^2 = E^2 = 1$.*

Before proceeding with a proof of the theorem, we establish some notation and terminology for the free product $G = *_{i\in I}G_i$. Our usage is that of Lyndon and Schupp [1] unless otherwise noted. A product PQ of elements P and Q of G is *reduced* if one of P, Q is trivial or if the last letter of the normal form of P is not inverse to the first letter of the normal form of Q. The product PQ is *fully reduced* if P or Q is trivial or if the last letter of the normal form of P is from a free factor different from that of the first letter of the normal form of Q; we sometimes denote this by writing $P \cdot Q$. These notions extend to products of more than two factors, with the understanding that the noncancellation conditions continue to apply after trivial factors have been deleted. Thus a product $P_1 \cdots P_k$ is fully reduced if and only if $|P_1 \cdots P_k| = \sum_{i=1}^{k} |P_i|$, where $|\ |$ denotes free product length.

An element P of G is *cyclically reduced* if $|P| \leq 1$ or the first and last letters of its normal form are not inverses and is *fully cyclically reduced* if $|P| \leq 1$ or the first and last letters of its normal form lie in different free factors of G.

A key ingredient in our analysis will be the characterization by Wicks of the fully reduced forms of a commutator in a free product. The following is a restatement of Lemma 6 of [3].

Lemma 4 (Wicks). *If $U \in G = *_{i\in I}G_i$ is a commutator, either $U \in W^{-1}G_iW$ for some $W \in G$, $i \in I$, and U is a commutator in $W^{-1}G_iW$, or some fully cyclically reduced conjugate of U has one of the following fully reduced forms:*

(4.1) *$X^{-1}a_1Xa_2$ with $X \neq 1$, $a_1 \neq 1$, $a_1, a_2 \in G_i$ for some $i \in I$, and a_1 conjugate to a_2^{-1} in G_i; or*

(4.2) *$X^{-1}a_1Y^{-1}a_2Xa_3Ya_4$ with $X \neq 1$, $Y \neq 1$, $a_1, a_2, a_3, a_4 \in G_i$ for some $i \in I$, and $a_4a_3a_2a_1 = 1$; or*

(4.3) *$X^{-1}a_1Y^{-1}b_1Z^{-1}a_2Xb_2Ya_3Zb_3$ with $a_1, a_2, a_3 \in G_i$ for some $i \in I$ and $a_3a_2a_1 = 1$, $b_1, b_2, b_3 \in G_j$ for some $j \in I$ and $b_3b_2b_1 = 1$, and either not all of $a_1, a_2, a_3, b_1, b_2, b_3$ are in any one free factor of G or each of X, Y, Z is nontrivial.*

As a final preliminary step, we examine the ways in which both an element and its inverse can occur as fully reduced subwords of a proper power in a free product.

Lemma 5. *Suppose that* V *is a fully cyclically reduced element of* $G = *_{i\in I} G_i$ *with* $|V| \geq 2$*, that* $m \geq 1$*, and that, for some* $X, R, S, T \in G$*,* $V^m = X^{-1} \cdot R = S \cdot X \cdot T$*. Then one of the following is true*:

(5.1) $|X| \geq |V|$, $X = X_1 \cdot B \cdot A$ *and* $V = A \cdot B$ *for some* A, B, X_1 *with* $A^2 = B^2 = 1$*, and* $SX = V^n \cdot A$ *for some* $n < m$*; or*

(5.2) $\frac{1}{2}|V| < |X| < |V|$, $X = X_1 \cdot X_2 \cdot X_3$ *and* $V = X_3 \cdot X_2^{-1} \cdot X_1 \cdot X_2$ *for some* X_1, X_2, X_3 *with* $X_1^2 = X_3^2 = 1$*, and* $S = V^n \cdot X_3 \cdot X_2^{-1}$ *for some* $n < m$*; or*

(5.3) $|X| < |V|$, $X = X_1 \cdot X_2$ *and* $V = X_2^{-1} \cdot X_1 \cdot X_2 \cdot T_1$ *for some* X_1, X_2, T_1 *with* $X_1^2 = 1$*, and* $S = V^n \cdot X_2^{-1}$ *for some* $n < m$*; or*

(5.4) $|X| < |V|$, $X = X_1 \cdot X_2$ *and* $V = X_2 \cdot X_1^{-1} \cdot S_2 \cdot X_1$ *for some* X_1, X_2, S_2 *with* $X_2^2 = 1$*, and* $S = V^n \cdot X_2 \cdot X_1^{-1} \cdot S_3$ *for some* $n < m$*; or*

(5.5) $|X| \leq \frac{1}{2}|V| - 1$ *and* $V = X^{-1} \cdot V_2 \cdot X \cdot V_3$ *for some nontrivial* V_2, V_3 *and* $S = V^n \cdot X^{-1} \cdot V_2$ *for some* $n < m$.

Proof of Lemma 5. If X is empty, clause (5.5) applies with $V = V_2 \cdot V_3$ a fully reduced factorization of V such that $S = V^n \cdot V_2$ for some $n < m$. We suppose, then, that $X \neq 1$.

If $|X| \geq |V|$, we factor V as $A \cdot B$ so that $SX = V^n \cdot A$ with $|A| < |V|$. It follows that $X = X_1 \cdot B \cdot A$ for some X_1. But since $X^{-1} = A^{-1} \cdot B^{-1} \cdot X_1^{-1}$ is an initial subword of $V^m = (A \cdot B)^m$, $A^{-1} = A$ and $B^{-1} = B$. This is the situation described in (5.1). We assume, henceforth, that $|X| < |V|$.

Let n be the largest integer such that $|V^n| \leq |S|$, and let S_1, V_1 be such that $S = V^n \cdot S_1$ and $V = X^{-1} \cdot V_1$. We cannot have $|S_1| = |X|$ or $|S_1| + |X| = |V|$, for that would violate our hypotheses on the fully reduced factorizations of V^m.

Suppose that $|S_1| < |X|$ and $|S_1| + |X| > |V|$. Then X factors as $X_1 \cdot X_2 \cdot X_3$ with $X^{-1} = S_1 \cdot X_1^{-1}$, $V = S_1 \cdot X_1 \cdot X_2$, and X_1 and X_2 nonempty. Now $S_1 = X_3^{-1} \cdot X_2^{-1}$, so $V = X_3^{-1} \cdot X_2^{-1} \cdot X_1 \cdot X_2$. But $SX = V^{n+1} \cdot X_3$, which implies that $X_3^{-1} = X_3$, and $V = X_3^{-1} \cdot X_2^{-1} \cdot X_1^{-1} \cdot V_1$, which yields $X_1^{-1} = X_1$. This is the situation of (5.2), and we note that $|V| < |S_1| + |X|$ and $|S_1| < |X|$ imply that $|V| < 2|X|$.

Next suppose that $|S_1| < |X|$ and $|S_1| + |X| < |V|$. Then X factors as $X_1 \cdot X_2$ with $S_1 = X_2^{-1}$ and $V = S_1 \cdot X \cdot T_1$ for some T_1, so $V = X_2^{-1} \cdot X_1 \cdot X_2 \cdot T_1 = X_2^{-1} \cdot X_1^{-1} \cdot V_1$. It follows that $X_1^{-1} = X_1$, and we are in situation (5.3).

Now suppose that $|S_1| > |X|$ and $|S_1| + |X| > |V|$. We factor X as $X_1 \cdot X_2$ with $V = S_1 \cdot X_1$ and factor S_1 as $X^{-1} \cdot S_3$. Then $V = X_2^{-1} \cdot X_1^{-1} \cdot S_3 \cdot X_1$ and, since $S \cdot X = V^{n+1} \cdot X_2$, $X_2^{-1} = X_2$; this is (5.4).

Finally, suppose that $|S_1| > |X|$ and $|S_1| + |X| < |V|$. In this case, S_1 factors as $X^{-1} \cdot V_2$ for some V_2 and $V = S_1 \cdot X \cdot V_3$ for some V_3. Then $V = X^{-1} \cdot V_2 \cdot X \cdot V_3$, where necessarily V_2 and V_3 are nonempty, and (5.5) applies. □

Proof of Theorem 1. Each of the forms specified for V (or, in (1.7), V^2) in the conclusion of Theorem 1 is preserved if V is replaced by a conjugate of itself, so we lose no generality in assuming that V is fully cyclically reduced. If $V \in G_i$ for some $i \in I$, then Lemma 4 tells us that (1.1) holds. We suppose, then, that $|V| \geq 2$.

By Lemma 4, some fully cyclically reduced conjugate of V^m has the form specified in (4.1), (4.2), or (4.3). After again replacing V by a fully cyclically reduced conjugate and relabeling in (4.2) and (4.3) if necessary, we may assume that V^m has form (4.1), or form (4.2) with $|X| \geq |Y|$, or form (4.3) with $|X| \geq |Y|$ and $|X| \geq |Z|$.

Let $P = a_1$ and $Q = a_2$ in form (4.1), $P = a_1 Y^{-1} a_2$ and $Q = a_3 Y a_4 = a_3 Y a_1^{-1} a_2^{-1} a_3^{-1}$ in form (4.2), and $P = a_1 Y^{-1} b_1 Z^{-1} a_2$ and $Q = b_2 Y a_3 Z b_3 = b_2 Y a_1^{-1} a_2^{-1} Z b_1^{-1} b_2^{-1}$ in form (4.3). In each instance, $V^m = X^{-1} \cdot P \cdot X \cdot Q$ and Q is conjugate to P^{-1} in G. Further, $|P| = |Q| = 1$ in (4.1), $|P| \leq |X| + 2$ and $|Q| \leq |X| + 2$ in (4.2), and $|P| \leq 2|X| + 3$ and $|Q| \leq 2|X| + 3$ in (4.3). We proceed by cases according to which clause of the conclusion of Lemma 5 is satisfied, with $R = PXQ$, $S = X^{-1}P$, and $T = Q$.

Case (5.1). Suppose that $X = X_1 \cdot B \cdot A$ and $V = A \cdot B$ for some X_1, A, B with $A^2 = B^2 = 1$, that $X_1^{-1} P X_1 = (AB)^k A$ for some k, $0 \leq k \leq m-3$, and that $Q = B(AB)^{m-k-3}$.

If m is even, (1.2) is satisfied, while if m is odd, Q conjugate to P^{-1} implies that B is conjugate to A and (1.3) holds.

Case (5.2). Suppose that $X = X_1 \cdot X_2 \cdot X_3$ and $V = X_3 \cdot X_2^{-1} \cdot X_1 \cdot X_2$ for some X_1, X_2, X_3 with $X_1^2 = X_3^2 = 1$, that $P = X_2 X_3 X_2^{-1} (X_1 X_2 X_3 X_2^{-1})^k$ for some k, $0 \leq k \leq m-3$, and $Q = X_2^{-1} X_1 X_2 (X_3 X_2^{-1} X_1 X_2)^{m-k-3}$.

As in the previous case, (1.2) applies if m is even, and if m is odd, Q conjugate to P^{-1} implies that X_3 is conjugate to X_1 and (1.3) obtains.

Case (5.3). Suppose that $|X| < |V|$, $X = X_1 \cdot X_2$ and $V = X_2^{-1} \cdot X_1 \cdot X_2 \cdot T_1$ for some X_1, X_2, T_1 with $X_1^2 = 1$, and that $P = X_2 T_1 X_2^{-1} (X_1 X_2 T_1 X_2^{-1})^k$ for some k, $0 \leq k \leq m-2$, and $Q = T_1 (X_2^{-1} X_1 X_2 T_1)^{m-k-2}$.

We first notice that since $|P| \leq 2|X| + 3 \leq 2|V| + 1$ and $|Q| \leq 2|X| + 3 \leq 2|V| + 1$, we have $m \leq 6$. Now Q is conjugate to P^{-1}, so P and Q must have fully cyclically reduced conjugates of the same length. It is not hard to see that this implies that either $k = m-k-2$ or $T_1^2 = 1$. If $T_1^2 = 1$, we find as in previous cases that (1.2) applies if m is even and that (1.3) applies if m is odd. We suppose, then, that $T_1^2 \neq 1$ and $k = m-k-2$. The possibilities to consider are that $m = 2$ and $k = 0$, $m = 4$ and $k = 1$, and $m = 6$ and $k = 2$.

If $m = 2$ and $k = 0$, T_1 is conjugate to T_1^{-1} and (1.6) holds.

If $m = 4$ and $k = 1$, $Q = T_1 X_2^{-1} X_1 X_2 T_1$ and $P = X_2 T_1 X_2^{-1} X_1 X_2 T_1 X_2^{-1}$, a conjugate of Q. Now $T_1^2 \neq 1$, so Q is not in a conjugate of a free factor of G, but since Q is conjugate to P^{-1}, Q is conjugate to Q^{-1}. By Lemma 3, then, $Q = DE$ for some D, E with $D^2 = E^2 = 1$. But then $V^2 = X_2^{-1} X_1 X_2 DE$, and (1.7) applies.

Suppose, then, that $m = 6$ and $k = 2$. We must have $|X| = |V| - 1$ and $|P| = |Q| = 2|V| + 1$, so X_2 is empty and T_1 has length one. Let us write $X_1 = C^{-1} \cdot a \cdot C$ with $C \in G$ and $a \in G_i$ for some $i \in I$ and $a^2 = 1$ and $T_1 = b \in G_j$ for some $j \in I$ with $b^2 \neq 1$. We then have $P = Q = b \cdot C^{-1} \cdot a \cdot C \cdot b \cdot C^{-1} \cdot a \cdot C \cdot b$, so $b^2 \cdot C^{-1} \cdot a \cdot C \cdot b \cdot C^{-1} \cdot a \cdot C$ is a fully cyclically reduced conjugate of P which, like P, is conjugate to its inverse. There must then be a factorization $C_1 \cdot C_2$ of C such that one of the following holds:

(1)
$$C_1^{-1}aC_1C_2b^{-1}C_2^{-1}C_1^{-1}aC_1C_2b^{-2}C_2^{-1}$$
$$= b^2C_2^{-1}C_1^{-1}aC_1C_2bC_2^{-1}C_1^{-1}aC_1C_2,$$

(2)
$$C_2b^{-1}C_2^{-1}C_1^{-1}aC_1C_2b^{-2}C_2^{-1}C_1^{-1}aC_1$$
$$= b^2C_2^{-1}C_1^{-1}aC_1C_2bC_2^{-1}C_1^{-1}aC_1C_2,$$

(3)
$$C_1^{-1}aC_1C_2b^{-2}C_2^{-2}C_1^{-1}aC_1C_2b^{-1}C_2^{-1}$$
$$= b^2C_2^{-1}C_1^{-1}aC_1C_2bC_2^{-1}C_1^{-1}aC_1C_2,$$

(4)
$$C_2b^{-2}C_2^{-1}C_1^{-1}aC_1C_2b^{-1}C_2^{-1}C_1^{-1}aC_1$$
$$= b^2C_2^{-1}C_1^{-1}aC_1C_2bC_2^{-1}C_1^{-1}aC_1C_2.$$

If (1) is true, a length comparison on the fully reduced products on the two sides shows that
$$C_1^{-1}aC_1C_2b^{-1}C_2^{-1} = b^2C_2^{-1}C_1^{-1}aC_1C_2$$
and
$$C_1^{-1}aC_1C_2b^{-2}C_2^{-1} = bC_2^{-1}C_1^{-1}aC_1C_2.$$
The left sides of these two equations begin with the same normal form letter, so looking at the right sides we get $b^2 = b$, a contradiction. Similarly, (2) yields
$$C_2b^{-1}C_2^{-1}C_1^{-1}aC_1 = b^2C_2^{-1}C_1^{-1}aC_1C_2$$
and
$$C_2b^{-2}C_2^{-1}C_1^{-1}aC_1 = bC_2^{-1}C_1^{-1}aC_1C_2,$$
from which we get the contradiction $b^2 = b$ if C_2 is nonempty or the equation $b^{-1} = b^2$ if C_2 is empty. This last possibility corresponds to (1.4). If (3) holds, we get
$$C_1^{-1}aC_1C_2b^{-2}C_2^{-1} = b^2C_2^{-1}C_1^{-1}aC_1C_2$$
and
$$C_1^{-1}aC_1C_2b^{-1}C_2^{-1} = bC_2^{-1}C_1^{-1}aC_1C_2.$$
As in (1), we derive the contradiction $b^2 = b$. Finally, if (4) is true,
$$C_2b^{-2}C_2^{-1}C_1^{-1}aC_1 = b^2C_2^{-1}C_1^{-1}aC_1C_2$$
and
$$C_2b^{-1}C_2^{-1}C_1^{-1}aC_1 = bC_2^{-1}C_1^{-1}aC_1C_2.$$
This yields the contradictions $b^{-1} = b$ if C_2 is empty and $b^2 = b$ if C_2 is nonempty.

Case (5.4). Suppose that $|X| < |V|$, $X = X_1 \cdot X_2$ and $V = X_2 \cdot X_1^{-1} \cdot S_2 \cdot X_1$ for some X_1, X_2, S_2 with $X_2^2 = 1$, that $P = S_2(X_1X_2X_1^{-1}S_2)^k$ for some k, $0 \le k \le m-2$, and $Q = X_1^{-1}S_2X_1(X_2X_1^{-1}S_2X_1)^{m-k-2}$.

Replacing V by its fully cyclically reduced conjugate $X_1X_2X_1^{-1}S_2$ and changing notation reduces this to Case (5.3).

Case (5.5). Suppose that $|X| \leq \frac{1}{2}|V| - 1$, $V = X^{-1} \cdot V_2 \cdot X \cdot V_3$ for some V_2, V_3, that $P = V_2(XV_3X^{-1}V_2)^k$ for some k, $0 \leq k \leq m-1$, and that $Q = V_3(X^{-1}V_2XV_3)^{m-k-1}$.

Since $|P| \leq 2|X| + 3 \leq |V| + 1$ and $|Q| \leq 2|X| + 3 \leq |V| + 1$, we have $m \leq 3$. We first consider the case that $m = 2$. If $k = 0$, $Q = V_3X^{-1}V_2XV_3$ conjugate to $P^{-1} = V_2^{-1}$ implies that $V_3^2 = 1$ and V_2 is conjugate to V_2^{-1}; (1.6) applies. If $k = 1$, $P = V_2XV_3X^{-1}V_2$ is conjugate to $Q^{-1} = V_3^{-1}$, so $V_2^2 = 1$, V_3 is conjugate to V_3^{-1}, and again (1.6) applies.

Now suppose that $m = 3$. In this event, we must have $|X| = \frac{1}{2}|V| - 1$ and $|P| = |Q| = |V| + 1$, so $|V_2| = |V_3| = 1$. Let us write $V_2 = a \in G_i$ for some $i \in I$ and $V_3 = b \in G_j$ for some $j \in I$. Then since $Q = b(X^{-1}aXb)^{2-k}$ is conjugate to $P^{-1} = a^{-1}(Xb^{-1}X^{-1}a^{-1})^k$, either $a^2 = b^2 = 1$ and a is conjugate to b, as described in (1.3), or $a^2 \neq 1$, $b^2 \neq 1$, $k = 1$, and there is a factorization $X_1 \cdot X_2$ of X such that one of the following holds:

$$X_2b^{-1}X_2^{-1}X_1^{-1}a^{-2}X_1 = b^2X_2^{-1}X_1^{-1}aX_1X_2, \tag{5}$$

$$X_1^{-1}a^{-2}X_1X_2b^{-1}X_2^{-1} = b^2X_2^{-1}X_1^{-1}aX_1X_2. \tag{6}$$

If (5) is true, either X_2 is empty and $a^3 = b^3 = 1$ as in (1.5) or X_2 is nonempty and $X_2b^{-1} = b^2X_2^{-1}$, so that $X_2 = b^2X_3$ and $X_2^{-1} = X_3^{-1}b^{-1}$ for some X_3, producing the contradiction $b^2 = b$. If (6) is true, $X_2^2 = 1$ and

$$X_1^{-1}a^{-2}X_1X_2b^{-1} = b^2X_2^{-1}X_1^{-1}aX_1.$$

If X_1 is nonempty, $X_1 = X_4b^{-1}$ and $X_1^{-1} = b^2X_4^{-1}$ for some X_4, whence $b^{-1} = b^{-2}$, a contradiction. Thus X_1 is empty, and $a^{-2}X_2b^{-1} = b^2X_2^{-1}a$ implies that $b^{-1} = a$ and $X_2 = X_2^{-1}$. Thus $V = XaXa^{-1}$ with $X^2 = 1$, and (1.3) applies. □

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Department of Mathematics, Eastern Illinois University, Charleston, Illinois 61920
E-mail address: cflpc@eiu.edu

Department of Mathematics, Mount St. Vincent University, Halifax, Nova Scotia, Canada B3M 2J6
E-mail address: cedmunds@linden.msvu.ca

Fachbereich Mathematik, Universität Dortmund, 4600 Dortmund 50, Germany
E-mail address: UMA004%DDOHRZ11.BITNET@vm.gmd.de